A MENSURATION

FOR

INDIAN SCHOOLS AND COLLEGES

PART I

A MENSURATION

FOR

INDIAN SCHOOLS AND COLLEGES

CONTAINING

NUMEROUS ILLUSTRATIVE EXAMPLES AND EXERCISES AND EXAMINATION QUESTIONS

DESIGNED TO MEET THE REQUIREMENTS OF UNIVERSITY, ENGINEERING COLLEGE, EUROPEAN SCHOOLS, ETC., EXAMINATIONS

PART I.
THE MENSURATION OF PLANE FIGURES

BY

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SOMETIME EXAMINER IN MATHEMATICS TO THE ALLAHABAD AND PUNJAB UNIVERSITIES

WITH ANSWERS

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PREFACE

IN preparing the following pages on Mensuration, I have been guided by the needs of Indian students. As a result, the book will be found to differ essentially from most existing text-books on the subject. Indian units have been introduced, numerous questions derived from previous Indian examination papers have been embodied, while the arrangement and scope of the subject have been carefully adapted to the courses of study prescribed by the various Indian examining bodies. I have found it possible to simplify the subject by arranging the material of each chapter on a uniform plan.

There is nothing more fatal to a beginner than the careless interpretation of a rule. If he is merely told to multiply half the height of a triangle by the base in order to find its area, he can hardly be expected to fully understand the *rationale* of the operation, while his difficulties are increased if these two measurements are expressed in terms of different units. To avoid this risk, I have made the statement of each rule so full and explicit as to leave no room for doubt, while, for memory purposes, I have added an abbreviated version. Only an elementary knowledge of Geometry and Algebra has been assumed in the proofs of formulæ.

Each chapter contains a series of Illustrative Examples

worked in full, and concludes with two sets of examples for solution by the student. In the first set the examples are original and graduated; in the second they are taken from previous examination papers.

The student is advised to omit the examination questions at the ends of chapters when reading through the book for the first time.

A. E. PIERPOINT.

CONTENTS

SYMBOLS

$+$	stands for	plus	$\therefore$	stands for	therefore
$-$	,,	minus	$\because$	,,	because
$\times$	,,	multiplied by	$\angle$	,,	angle
$\div$	,,	divided by	$\triangle$	,,	triangle
$\sqrt{}$	,,	square root of	$>$	,,	is greater than
$=$	,,	is equal to	$<$	,,	is less than

ELEMENTARY MENSURATION

PART I.

CHAPTER I.

INTRODUCTORY—TABLES.

1. MENSURATION is the science which investigates the lengths of lines, the areas of surfaces, and the volumes of solids.

2. In order to measure a magnitude of any kind, it must be compared with and expressed in terms of some fixed magnitude of its own kind, which is chosen arbitrarily, and is called a *unit.* For example, the *foot* unit may be used in the measurement of length, the *pound* unit in the measurement of weight.

3. It is convenient to have a set of units for the measurement of each kind of magnitude. For example, if we are measuring an area which is very extensive, we shall find the *acre* unit or the *square mile* unit more suitable than the *square inch* unit or the *square foot* unit.

4. We shall often find it convenient to speak of *corresponding* units of length, area, and volume. If we take any unit of length and on it describe a square, then the area of this square will be that unit of area that *corresponds* to the given unit of length. And if a cube be constructed on this square as base, then the volume of this cube will be that unit of volume that *corresponds* to the given unit of length.

Thus the cubic foot and the square foot *correspond* to the linear foot.

5. The units of length and area most commonly used in mensuration are given in the following tables :—

Linear Measure (English).

12 inches make 1 foot.
3 feet make 1 yard.
5½ yards make 1 rod, pole, or perch.
40 poles make 1 furlong.
8 furlongs, or 1760 yards, make 1 mile.
3 miles make 1 league.

Square Measure (English).

144 square inches make 1 square foot.
9 square feet make 1 square yard.
$30\frac{1}{4}$ square yards make 1 square rod, pole, or perch.
40 square poles make 1 rood.
4 roods, or 4840 square yards, make 1 acre.
640 acres make 1 square mile.

In the measurement of land, the unit most frequently used is a chain 22 yards long, consisting of 100 links, which is called *Gunter's Chain.*

Thus we have, as an addition to the above tables—

100 links, or 22 yards, make 1 chain;

and

10,000 square links make 1 square chain.
10 square chains make 1 acre.

Linear Measure (British Indian).

8 girahs make 1 hath.
2 haths „ 1 guz.
$2\frac{1}{2}$ guz „ 1 latha.
20 lathas or gathas, or 55 English yards, make 1 rasi or jarib.

Square Measure (British Indian).

20 biswansi make 1 biswa.
20 biswas, or 1 square rasi, make 1 bigha.

Hence 1 bigha $= (55 \times 55)$ English square yards
$= 3025$ English square yards
$= \frac{5}{8}$ of an acre.

However, the bigha varies considerably in area in different districts.

ILLUSTRATIVE EXAMPLES.

6. *Example* 1.—How many yards are equivalent to a length of 3 mi. 6 fur. 20 po.?

$$\begin{array}{r} 3 \text{ mi. } 6 \text{ fur. } 20 \text{ po.} \\ 8 \\ \hline 30 \text{ fur.} \\ 40 \\ \hline 1220 \text{ po.} \\ 5\frac{1}{2} \\ \hline 6100 \\ 610 \\ \hline 6710 \text{ yds.} \end{array}$$

Example 2.—Reduce 325608 in. to miles.

12)325608 in.
3)27134 ft. 0 in.
9044 yds. 2 ft.
2
11)18088 half-yds.
40)1644 po. 4 half-yds.
8)41 fur. 4 po.
5 mi. 1 fur.

Hence—

325608 in. = 5 mi. 1 fur. 4 po. 2 yds. 2 ft.

Example 3.—How many square feet are there in 3 ac. 2 ro. 14 sq. po. 10 sq. yds.?

3 ac. 2 ro. 14 po. 10 yds.
4
14 ro.
40
574 sq. po.
$30\frac{1}{4}$
17230
$143\frac{1}{2}$
$17373\frac{1}{2}$ sq. yds.
9
$156361\frac{1}{2}$ sq. ft.

Example 4.—Reduce 483260 sq. ft. to acres.

9)483260 sq. ft.
53695 sq. yds. 5 sq. ft.
4
121 { 11)214780
11)19525—5 }
40)1775—0 } 5 quarter sq. yds. = $1\frac{1}{4}$ sq. yd
4)44 ro. 15 sq. po.
11 ac.

∴ 483260 sq. ft. = 11 ac. 15 sq. po. $1\frac{1}{4}$ sq. yd. 5 sq. ft.

Examples—I. A.

1. Reduce 3 fur. 15 po. 4 yds. to yards.
2. Reduce 2 mi. 7 fur. $4\frac{1}{2}$ yds. to feet.
3. Express 5 mi. 6 fur. 19 po. 3 yds. as feet.
4. How many chains are there in 6 mi.?
5. Reduce 3 ac. 2 ro. to square poles.
6. Reduce 4 ac. 3 ro. 14 sq. po. to square yards.

7. Express 9 ac. as square links.
8. How many miles, furlongs, etc., are there in 98734 ft.?
9. Reduce 738629 sq. lks. to acres.
10. Reduce 894671 sq. ft. to acres, roods, etc.
11. What is the measure of 3 fur. 20 po. when a line measuring 5 yds. 1 ft. 6 in. is taken as the unit?
12. What is the measure of $3\frac{1}{2}$ ac. when an area measuring 8 sq. yds. is taken as the unit?
13. What is the unit of length when a distance of 2 mi. measures 220?
14. What is the unit of area when a field of 5 ac. measures 40?

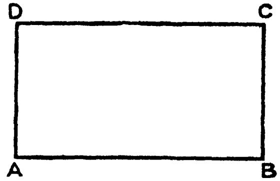
Examples—I. B.

15. Reduce 2 lathas 2 guz to girahs.
16. Express 400 girahs as lathas.
17. How many haths are there in 17·5 rasis?
18. Reduce 385 English yards to girahs.
19. Find the number of biswansi in 17 bighas.
20. Reduce 5321 biswansi to bighas, biswas, etc.
21. How many biswansi are there in an acre?
22. Reduce 2 ac. 3 ro. 30 sq. po. to biswas.

CHAPTER II.

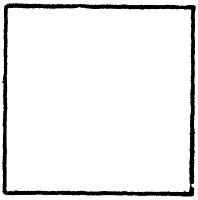
ON RECTANGLES.

7. A *rectangle* is a four-sided figure having all its angles right angles.



The length and breadth of a rectangle are called its *dimensions*. Thus AB and BC are the *dimensions* of the rectangle $ABCD$.

When the dimensions of a rectangle are equal to one another, the figure is called a *square* (see fig.).

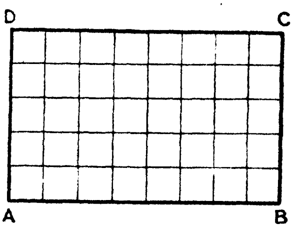


The *perimeter* (or periphery) of a figure is the sum of its boundaries.

PROPOSITION I.

8. *To find the area of a rectangle, having given its dimensions.*

Let $ABCD$ be the plan of a rectangular room, such that AB represents a length of 8 yds., and BC a length of 5 yds. It is required to find the area of the room.



Divide AB into eight equal parts, and BC into five equal parts, so that each of these parts will represent a yard.

Through the points of division in AB draw straight lines parallel to BC, and through the points of division in BC draw straight lines parallel to AB. The rectangle is now divided into five rows, each row containing eight squares, and each square representing a square yard.

Note that the number of rows is the same as the number of yards in BC, and the number of squares in each row is the same as the number of yards in AB.

$\therefore$ the rectangle contains $8 \times 5 = 40$ squares, and each square represents a square yard.

$\therefore$ the rectangular room measures 40 sq. yds.

From this special case we may arrive at the general conclusion—

If one of the dimensions of a rectangle measure a of any unit of length, and if the other dimension measure b *of the same unit of length*, then the area of the rectangle will measure $a \times b$ of the *corresponding unit of area.*

Hence rule—

The number of any linear unit in the length of a rectangle multiplied by the number of the same linear unit in the breadth gives the number of the corresponding square unit in the area.

Or briefly—

$$\textbf{area of rectangle} = \textbf{length} \times \textbf{breadth}$$

$$\mathbf{A} = \mathbf{a} \times \mathbf{b} \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

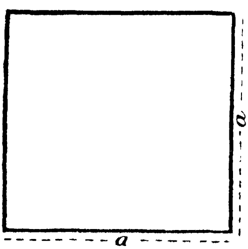
$$\therefore \textbf{length of rectangle} = \frac{\textbf{area}}{\textbf{breadth}}$$

$$\mathbf{a} = \frac{\mathbf{A}}{\mathbf{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$\textbf{and breadth of rectangle} = \frac{\textbf{area}}{\textbf{length}}$$

$$\mathbf{b} = \frac{\mathbf{A}}{\mathbf{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

PARTICULAR CASE.



9. Square.

Here the dimensions are equal to one another. That is—

$$\text{length} = \text{breadth} = \text{side}$$

Now, area of any rectangle = lgth × brdth § 8.

$$\therefore \text{area of square} = \text{side} \times \text{side}$$

$$= (\text{side})^2$$

$$A = a^2$$

$$\therefore \text{side of square} = \sqrt{\text{area}}$$

$$a = \sqrt{A}$$

Hence rule—

The square root of the number of any square unit in the area of a square gives the number of the corresponding linear unit in a side.

Or briefly—

$$\textbf{side of square} = \sqrt{\textbf{area}}$$

$$a = \sqrt{A}$$

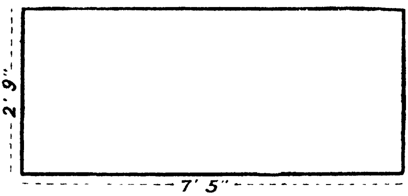
10. The student must be careful to always bear in mind the correct interpretation of these abbreviated statements. When we say that—

$$\frac{\text{area}}{\text{breadth}} = \text{length of rectangle}$$

we appear to be asserting that an area when divided by a breadth gives a length. This, of course, is absurd. What we are really asserting is that, by dividing the number of any square unit in the area of a rectangle by the number of the corresponding linear unit in the breadth, we obtain the number of the corresponding linear unit in the length. The same caution must be observed when interpreting most of the formulæ in mensuration.

ILLUSTRATIVE EXAMPLES.

11. *Example* 1.—Find the area of a rectangle whose length is 7 ft. 5 in. and breadth 2 ft. 9 in.



Area of rect. $= (a \times b)$ sq. in. . . . . § 8.

where $a = (7 \times 12 + 5) = 89$,
and $b = (2 \times 12 + 9) = 33$;

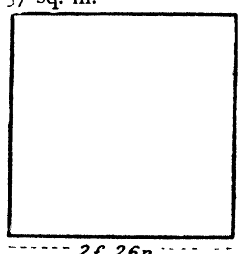
$\therefore$ area of rect. $= 89 \times 33$ sq. in.
$= 2937$ sq. in.
$= 2$ sq. yds. 2 sq. ft. 57 sq. in.

Example 2.—Find the area of a square field whose side measures 2 fur. 26 po.

Area of square $= a^2$ sq. po. . . § 9.

where $a = (2 \times 40 + 26) = 106$;

$\therefore$ area of field $= (106)^2$ sq. po.
$= 11236$ sq. po.
$= 70$ ac. 0 ro. 36 sq. po.



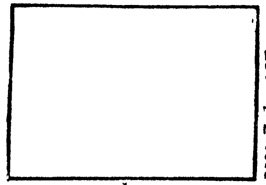
Example 3.—Find the rent of a rectangular field whose length is 10 ch. 25 lks., and whose breadth is 7 ch. 45 lks., at Rs.30 per acre.

Area of field $= (a \times b)$ sq. ch. . § 8.

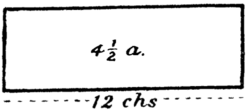
where $a = 10{\cdot}25$,
and $b = 7{\cdot}45$;

$\therefore$ area of field $= (10{\cdot}25 \times 7{\cdot}45)$ sq. ch.
$= 76{\cdot}3625$ sq. ch.
$= 7{\cdot}63625$ ac.

$\therefore$ rent $=$ Rs.$30 \times 7{\cdot}63625$
$=$ Rs.$229{\cdot}0875$
$=$ Rs.229 1 anna 4·8 pies



Example 4.—Find the breadth of a rectangle whose area is $4\frac{1}{2}$ ac., and whose length is 12 ch.

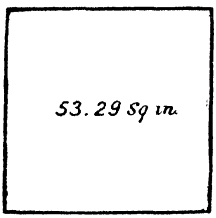


Breadth of rect. $= \frac{A}{a}$ ch. . . . § 8.

where $A = 4\frac{1}{2} \times 10 = 45$,
and $a = 12$;

$\therefore$ breadth of rect. $= \frac{45}{12}$ ch.
$= 3{\cdot}75$ ch.
$= 3$ ch. 75 lks.

Example 5.—Find the perimeter of a square whose area measures $53{\cdot}29$ sq. in.



One side of the square $= \sqrt{A}$ in. . § 9.

where $A = 53{\cdot}29$;

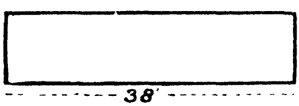
$\therefore$ one side of the square $= \sqrt{53{\cdot}29}$ in.
$= 7{\cdot}3$ in.

and the perimeter = one side $\times$ 4

$\therefore$ the perimeter $= (7{\cdot}3 \times 4)$ in.
$= 29{\cdot}2$ in.

Example 6.—The cost of painting a rectangular wall is Rs.5 15 annas, at 2 annas 6 pies per square yard. If the wall is 38 ft. long, find its height.

area of wall = (Rs.5 15 annas $\div$ 2 annas 6 pies) sq. yds.
$= (95 \div 2\frac{1}{2})$ sq. yds.
$= 38$ sq. yds.



Now, height of wall $= \frac{A}{a}$ ft. . . . § 8.

where $A = 38 \times 9$,
and $a = 38$;

$\therefore$ height of wall $= \frac{38 \times 9}{38}$ ft.
$= 9$ ft.

Example 7.—A room is 21 ft. long, 16 ft. wide, and 11 ft. high. In it there is a door 7 ft. by 3 ft., and two windows 8 ft. by 4 ft. Find the cost of papering the walls with paper 2 ft. wide, at $2\frac{1}{2}$ annas a yard.

area of two side walls $= 2 \times 21 \times 11$ sq. ft. . . . . § 8.
area of two end walls $= 2 \times 16 \times 11$ sq. ft. . . . . § 8.
area of door $= 7 \times 3$ sq. ft. . . . . § 8.
area of two windows $= 2 \times 8 \times 4$ sq. ft. . . . . § 8.

and total area to be covered with paper $\}$ = 4 walls − one door − two windows

$\therefore$ total area to be covered with paper $\} = (2 \times 21 \times 11 + 2 \times 16 \times 11 - 7 \times 3 - 2 \times 8 \times 4)$sq.ft.
$= 729$ sq. ft.

Now, paper is sold in rectangular strips;

$\therefore$ required length of paper $= \frac{\text{required area of paper}}{\text{width of paper}}$ . . . § 8.

$$= \tfrac{729}{2} \text{ ft.}$$
$$= 364\tfrac{1}{2} \text{ ft.}$$

$\therefore$ cost of paper at $2\frac{1}{2}$ annas a yard, or $\frac{5}{8}$ anna a foot $\Big\} = 364\frac{1}{2} \times \frac{5}{8}$ annas

$$= \text{Rs.}18 \ 15 \text{ annas } 9 \text{ pies}$$

Example 8.—The area of a rectangular courtyard is 2400 sq. yds., and its sides are in the ratio of 3 to 2: find the cost of fencing it at the rate of 4 annas a foot.

Let x yds. = length of courtyard.

$$\text{Then } \frac{2x}{3} \text{ yds.} = \text{breadth}$$

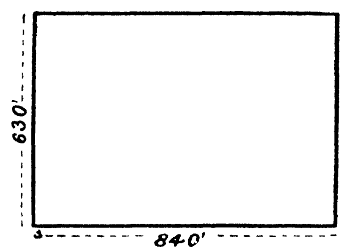
$$\therefore \text{area of courtyard} = \frac{2x^2}{3} \text{ sq. yds.} \quad \ldots\ldots \quad \S\ 8.$$
$$= 2400 \text{ sq. yds.}$$
$$\therefore x^2 = 3600$$
$$x = 60$$

Thus length of courtyard = 60 yds.
and breadth of courtyard = 40 yds.

Hence—

perimeter of courtyard = 200 yds.
$\therefore$ cost of fencing = 200×12 annas
= Rs.150

Example 9.—Find the cost of paving a rectangular enclosure, 840 ft. long and 630 ft. broad, with stones each 4 ft. 8 in. by 4 ft. 6 in., at Rs.4 a hundred.

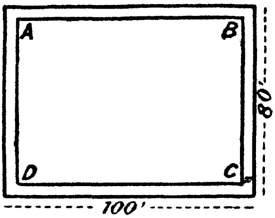


$$\text{Area of enclosure} = 840 \times 630 \text{ sq. ft.} \quad \ldots\ldots \quad \S\ 8.$$
$$\text{area of each stone} = 4\tfrac{2}{3} \times 4\tfrac{1}{2} \text{ sq. ft.} \quad \ldots\ldots \quad \S\ 8.$$

$$\therefore \text{number of stones required} = \frac{840 \times 630}{4\frac{2}{3} \times 4\frac{1}{2}}$$

$$\text{hence cost at Rs.4 a hundred} = \text{Rs.}\frac{840 \times 630 \times 4}{4\frac{2}{3} \times 4\frac{1}{2} \times 100}$$
$$= \text{Rs.}1008$$

* *Example* 10.—There is a garden 100 ft. long and 80 ft. broad, and a gravel walk is to be made of an equal width all round it, so as to take up one-fifth of the garden: what must be the breadth of the walk?



Let x feet = breadth of walk.

Then area of rect. $ABCD$ $= (100 - 2x) \times (80 - 2x)$ sq. ft. . . § 8.

but area of garden $= 100 \times 80$ sq. ft. § 8.

$\therefore$ area of walk $= \{100 \times 80 - (100 - 2x) \times (80 - 2x)\}$ sq. ft.

$= (360x - 4x^2)$ sq. ft.

But area of walk = one-fifth area of garden

$$\begin{aligned} \therefore 360x - 4x^2 &= 1600 \\ x^2 - 90x &= -400 \\ x^2 - 90x + (45)^2 &= 2025 - 400 \\ &= 1625 \\ x - 45 &= \pm 40{\cdot}311 \ldots \\ x &= 4{\cdot}68 \ldots \end{aligned}$$

hence breadth of walk = 4·68 ft. nearly

Examples—II. A.

Find the areas in square feet of rectangles having the following dimensions:—

1. Length 13 ft., breadth 11 ft.
2. Length 27 ft., breadth 19 ft.
3. Length 73 ft., breadth 23 ft.

Find the areas in square feet and square inches of rectangles having the following dimensions:—

4. Length 2 ft. 9 in., breadth 1 ft. 7 in.
5. Length 3 ft. 9 in., breadth $2\frac{1}{3}$ ft.
6. Length 5 ft. 8 in., breadth 2 ft. 10 in.

Find the areas in square yards, square feet, and square inches of rectangles having the following dimensions:—

7. Length 2 yds. 2 ft. 9 in., breadth 1 yd. 6 in.
8. Length 4 yds. 1 ft. 4 in., breadth 2 yds. 2 ft. 3 in.
9. Length 5 yds. 10 in., breadth 3 yds. 1 ft. 7 in.

Find the areas in acres of rectangles having the following dimensions:—

10. Length 26 ch., breadth 18 ch.
11. Length 12 ch. 16 lks., breadth 9 ch.
12. Length 13 ch. 43 lks., breadth 10 ch. 18 lks.

Find the areas in acres, roods, poles, of rectangles having the following dimensions:—

13. Length 7 ch. 15 lks, breadth 3 ch.
14. Length 10 ch. 22 lks., breadth 6 ch. 14 lks.
15. Length 8 ch. 40 lks., breadth 7 ch. 90 lks.

Find the length in each of the rectangles having the following measurements:—

16. Area 3 sq. yds., breadth 1 yd. 1 ft.
17. Area 1 ro., breadth 10 yds.
18. Area 1 ac., breadth 40 yds.
19. Area $7\frac{1}{2}$ ac., breadth 5 ch.
20. Area 59 sq. yds. 1 sq. ft. 2 sq. in., breadth 6 yds. 1 ft. 7 in.

Find the breadth in each of the rectangles having the following measurements:—

21. Area 24 ac., length 16 ch.
22. Area 2 ac., length 110 yds.
23. Area 6·35 ac., length 1000 lks.
24. Area 288 ac. 3 ro. 21 po., length 5 fur. 17 po.

Find the areas in square yards and square feet of squares having the following measurements:—

25. Side 9 yds. 1 ft.
26. Side 7 yds. 2 ft.
27. Side 12 yds. 2 ft.
28. Side 16 yds. 1 ft.

Find the areas in acres of squares having the following measurements:—

29. Side 15 ch.
30. Side 13 ch. 29 lks.
31. Side 6 ch. 84 lks.
32. Side 11 ch. 94 lks.

Find the side of each of the squares having the following areas:—

33. Area 324 sq. ft. (give the result in feet).
34. Area 1·225 ac. (give the result in chains).
35. Area 54·756 ac. (give the result in chains).
36. Area 4 ac. 2 sq. ch. 5104 sq. lks. (give the result in chains and links).

37. Find the cost of paving a rectangular courtyard 36 yds. long and 28 yds. wide, at the rate of Rs.1 12 annas per square yard.

38. Find the cost of carpeting a room 18 ft. long and 14 ft. 9 in. wide at the rate of Rs.2 per square yard.

39. What is the area in square yards and square feet of the walls of a rectangular room 25 ft. long, 17 ft. wide, and 11 ft. high?

40. What length of carpet 18 in. wide is required to cover a floor whose area is 42 sq. yds.?

41. The width of a rectangle is one-third its length: find its area if its length is 32 ch.

42. The perimeter of a square is 5 yds. 1 ft. 8 in.: find its area in square yards, square feet, and square inches.

43. The area of a square is 1·05625 ac.: find its perimeter in chains.

44. Find the cost of putting a fence round a square whose area is 160 sq. yds. 4 sq. ft., at the rate of Rs.2 8 annas per foot.

45. If the cost of tiling a floor is Rs.246, at the rate of 12 annas per square foot, and if the length of the floor is 8 yds., find its breadth.

46. The rent of a rectangular piece of ground is Rs.500, at the rate of Rs.12 8 annas per acre: find its dimensions in chains, if its length is four times its breadth.

47. If the cost of a fence round a square enclosure is Rs.208, at the rate of Rs.3 4 annas per yard, find the area in square yards.

48. How many yards of paper 30 in. wide are required for a wall 18 ft. 9 in. long and 16 ft. 6 in. high?

49. Find the cost of carpeting a room 26 ft. long by 21 ft. broad with carpet 1 ft. 6 in. wide at Rs.3 8 annas a yard.

50. A courtyard is 12 yds. 2 ft. long and 10 yds. 1 ft. wide: find the cost of paving it with bricks 9 in. long and 4 in. wide, if the bricks cost 8 annas a dozen.

Examples—II. B.

Find the areas in bighas of rectangles having the following dimensions:—

51. Length 7 rasi, breadth 5 rasi.

52. Length 3·8 rasi, breadth 2·4 rasi.

53. Length 121 yds., breadth 50 yds.

Find the length of each of the rectangles having the following measurements:—

54. Area 3 bighas, breadth 10 lathas.

55. Area 5 bighas 10 biswas, breadth 2 rasi.

56. Area 4 bighas 4 biswas 10 biswansi, breadth 80 yds.

Examination Questions—II.

A. *Allahabad University: Matriculation.*

1. A room, whose length is 30 ft. and breadth twice its height, takes 144 yds. of paper 2 ft. wide for its four walls: find the area of the floor.

2. A rectangular field of 5 ac., 200 yds. long, is planted with trees in rows perpendicular to the length, one yard from row to row, and one yard from tree to tree in the same row. If a width of a yard all round the field remain unplanted, find the number of trees.

3. Find the cost of lining a rectangular cistern 12 ft. 9 in. long, 8 ft. 3 in. broad, and 6 ft. 6 in. deep, with sheet lead weighing 8 lbs. per square foot, and which cost £1 8*s*. per cwt.

B. *Punjab University: Middle School.*

4. Find how many students can sit in a room measuring 20 yds. by 28 ft., supposing each student to require a space of 4 ft. by 30 in.

C. *Calcutta University: Matriculation.*

5. A room is 34 ft. long, $18\frac{1}{2}$ ft. wide, and 12 ft. high: find the expense of papering the walls at 1*s*. 6*d*. per square yard.

D. *European Schools: Final. United Provinces.*

6. Find in square feet the total area of the walls, floor, and ceiling of a room $22\frac{1}{2}$ ft. long, $16\frac{1}{2}$ ft. wide, and $13\frac{1}{3}$ ft. high.

7. A plank is 18 in. wide: find what length must be cut off that the area may be a square yard.

E. *Roorkee Engineer: Entrance.*

8. Find the expense of carpeting a room 26 ft. long and 18 ft. broad, with carpet 27 in. wide at 4*s*. 8*d*. per yard.

9. A square field contains 31 ac. 0 ro. 10·25 sq. po.: find the length of a side.

10. The length and breadth of a rectangular enclosure are 47 yds. 2 ft. 4 in. and 22 yds. 2 ft. 11 in.: what should be the breadth of another rectangular enclosure if its length is 63 yds. 1 ft. 5 in., and its area $\frac{5}{8}$ of the former?

11. A rectangular grass-plot, the sides of which are as 2 : 3, cost £14 8*s.* for turfing, at the rate of 4*d.* per square yard : find the lengths of its sides.

12. The area of the two side walls of a rectangular room is 806 sq. ft., the area of the two end walls 546 sq. ft. : find the dimensions of the room.

13. Find the cost of lining a rectangular cistern with lead at Rs. 1 2 annas for each square foot of surface, the inside dimensions of the cistern being as follows, the length 3 ft. 2 in., the breadth 2 ft. 10 in., and the depth 2 ft. 6 in.

14. Two square fields jointly contain 6 ac., and the side of one is three-fourths as long as that of the other : how many acres in each ?

15. There is a garden 140 ft. long and 120 ft. broad, and a gravel walk is to be made of an equal width all round it, so as to take up just one-fourth of the garden : what must be the breadth of the walk ?

F. *Roorkee Upper Subordinate : Entrance.*

16. A country in the form of a rectangle, 600 miles long by 200 miles broad, supports a population of 20,000,000 : find the average number of acres required to support one person.

17. The breadth of a rectangular room is two-thirds its length, and it costs £39 8*s.* 8*d.* to cover the floor with carpet 27 in. wide at 5*s.* 3*d.* a yard. To paper the walls costs £2 3*s.* 4*d.* with paper 21 in. wide, at 2*s.* 4*d.* per piece of 12 yds. : find the height of the room.

18. In the centre of a room, which is 19 ft. $1\frac{1}{2}$ in. square, a square carpet is placed, the rest of the floor being covered by a parquet border of uniform width, which is charged for at the rate of $7\frac{1}{2}$*d.* per square foot. If the carpet is charged for at $10\frac{1}{2}$*d.* per square foot, and the whole cost of carpet and parquet is £14 13*s.* $3\frac{3}{4}$*d.*, find the width of the parquet border.

19. There are two rectangular rooms of the same height, one is 19 ft. by 14 ft., the other 17 ft. by 15 ft. To cover the walls with paper 27 ins. wide at 3*s.* 9*d.* per piece of 12 yds. costs £3 12*s.* $2\frac{3}{4}$*d.* : find the height of the rooms.

20. Two square rooms, one 2 ft. longer each way than the other, are of equal height, and cost respectively £3 14*s.* 9*d.* and £3 8*s.* 3*d.* to paper the walls at $6\frac{1}{2}$*d.* per square yard : find the height.

21. How many planks of $1\frac{1}{2}$ in. in thickness can be cut out of a piece of timber 21 in. thick, allowing $\frac{1}{8}$ in. for each saw-cut ?

22. The area of a square is $22\cdot\dot{2}$: find the side of a square of half the size.

23. How many planks, 10 ft. long and 8 in. broad, will be required for the floor of a room whose length is 30 yds. and breadth 12 yds. ?

24. How long will it take to walk round a square field containing 13 ac. 1089 yds., at the rate of $2\frac{1}{4}$ miles an hour ?

25. The cost of a square field, at £2 14*s.* 6*d.* per acre, amounts to £27 5*s.* : find the cost of putting a paling round the field at 9*d.* per yard.

26. What is the difference between the superficial contents of a floor, 28 ft. long and 20 broad, and that of two others only half its dimensions ?

G. *Roorkee Engineer : Final.*

27. The perimeter of one square is 748 in., and that of another is 336 in. : find the perimeter of a square which is equal in area to the other two.

28. The length of a room is double the breadth ; the cost of colouring the ceiling, at $4\frac{1}{2}$*d.* per square yard, is £2 12*s.* 1*d.*, and the cost of painting the four walls, at 2*s.* 4*d.* per square yard, is £35 : find the height of the room.

29. What length of matting, $\frac{3}{4}$ of a yard wide, will be required for a room 15 ft. 8 in. long by 11 ft. 3 in. wide ; and what will be the cost at 6 annas per yard ?

30. The area of a rectangular courtyard is 2000 sq. yds., and its sides are in the ratio of 1·25 to 1. A pavement of uniform width runs along the four sides of the courtyard, and occupies half its area : find the width of the pavement.

31. Find how many yards of paper will be required for the walls of a room which is 24 ft. long, 19 ft. 6 in. broad, and 14 ft. high, the paper being $\frac{3}{4}$ yd. wide.

32. A box, without a lid, and made of wood 1 in. thick, requires painting inside and out; its exterior length, breadth, and depth are 3, 2, and $1\frac{1}{2}$ ft. respectively: how many superficial feet of paint will be required for each coat?

II. *Roorkee Upper Subordinate: Monthly.*

33. Find how many trees there are in a wood, 1 mi. long and a quarter of a mile wide, supposing, on an average, four trees grow on each square chain.

34. In a rectangular garden, 120 ft. long and 90 ft. broad, a walk passing round it, with its outer edge 10 ft. from the wall, occupies a fourth part of the garden: what is the width of the walk?

35. How much paper, $\frac{5}{13}$ of a yard broad, will be required to paper a room which is 22 ft. by 20 ft., and 13 ft. high?

36. A square field of grain, containing 10 ac., is to be cut by a reaper working round and round. The cut of the reaper is 5 ft.: how many rounds must the reaper take to cut three-fourths of the field?

37. The length of a railway is $47\frac{1}{2}$ mi., and the average breadth of land required for its formation 57 yds.: what will be the amount of purchase of the land at £50 per acre?

38. A rectangular enclosure is 120 ft. long and 70 broad; a walk of uniform width is made round the outside of it equal in area to the enclosure: find the width of the walk.

I. *Roorkee Upper Subordinate: Final.*

39. A room measures 28 ft. by 16 ft. In the centre is a Turkey carpet 24 ft. by 12 ft.: how much oil cloth would be required to cover the remainder of the floor, supposing the oil cloth to be 20 in. wide?

40. Find what length of wall paper, 27 in. wide, will be required for a room 20 ft. long, 16 ft. broad, and $10\frac{1}{2}$ ft. high. In it are two windows 6 ft. by 4 ft., a door 7 ft. by 4 ft., and a fireplace 4 ft. by 3 ft. 6 in.

41. Find how many planks, 12 ft. 6 in. long by $9\frac{1}{2}$ in. wide, will be required to floor a room 40 ft. by 20 ft.

J. *Additional Examination Questions.—II.* (For Answers, see p. 167.)

42. If 11,000 copies of the *Times* be issued daily, each copy consisting of two sheets, and each sheet being 4 ft. by 3 ft., how many acres will one edition cover? (Punjab University: First Examination in Civil Engineering.)

43. A garden is 160 ft. long and 120 ft. broad; there is a tank in the garden, leaving a space of equal width round it, and occupying half the area of the garden: find the length and breadth of the tank. (Roorkee Upper Subordinate: Monthly.)

44. The area of a rectangular field is 15 acres, and its length is half as much again as its breadth. How long will it take a man to walk four times round it at the rate of 3 miles an hour? (European Schools: Final. U.P.)

45. A room 18 ft. long, 15 ft. wide, and 12 ft. high, contains two doors 7 ft. by 4 ft., and (4 ft. from the ground) two windows 4 ft. by 3 ft.; a dado $2\frac{1}{2}$ ft. high runs round the room: find the cost of papering the walls at 1 anna per square foot. (European Schools: Final. U.P.)

46. A building has 63 windows; 40 of them contain 12 panes each $20'' \times 16''$, the others contain 9 panes each 16 in. square: find the cost of glazing the whole at Rs. 2 per square foot. (Roorkee Engineer: Final.)

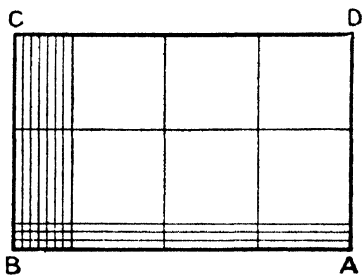
CHAPTER III.

ON DUODECIMALS.

12. WHEN the dimensions of a rectangle are given as compound quantities (that is, expressed in terms of two or more units), we have found it necessary to express them both as simple quantities in terms of the same unit, in order to find the area of the rectangle.

This reduction may be avoided by the use of *duodecimals*.

13. Consider the rectangle $ABCD$.



Let its length AB be taken to represent 3 ft. 7 in., and its breadth BC 2 ft. 3 in.

On the same scale, from A along AB, divide off three lengths each corresponding to one foot, and seven lengths each corresponding to one inch; and from C along CB divide off two lengths each corresponding to one foot, and three lengths each corresponding to one inch.

Through the points of division draw straight lines parallel to BC and AB respectively. We now see that the area of the rectangle is made up of several pieces of three different sizes.

The largest pieces represent squares measuring one foot each way (*i.e.* square feet), and of these there are 3×2.

The smallest pieces represent squares measuring one inch each way (*i.e.* square inches), and of these there are 7×3.

The remaining pieces represent rectangles measuring one foot by one inch, which we shall call *superficial primes*, and of these there are $(7 \times 2 + 3 \times 3)$.

Thus the rectangle $ABCD$ represents an area equal to the sum of—

(1) 3×2 square feet $=$ 6 square feet
(2) $(7 \times 2 + 3 \times 3)$ superficial primes $=$ 23 superficial primes
(3) 7×3 square inches $=$ 21 square inches

and because a superficial prime is a rectangle measuring one foot by one inch, it is evident that

12 square inches = 1 superficial prime;
12 superficial primes = 1 square foot.

It will be found that the above result may be obtained by the following scheme of work, in which we multiply each term in one dimension of the rectangle by each term in the other dimension.

3 ft.	7 in.	
2 ft.	3 in.	
3×2 sq. ft.	7×2 superficial primes	
	3×3 " "	7×3 sq. in.
6 sq. ft.	23 superficial primes	21 sq. in.

and because 12 sq. in. = 1 superficial prime
and 12 superficial primes = 1 sq. ft.

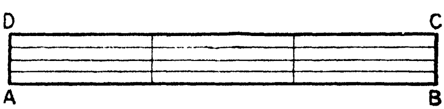
this result may be written—

8 sq. ft. 0 superficial primes 9 sq. in.

In this scheme of work it is seen that 7 inches, when multiplied by 2 feet, are assumed to give 7×2 superficial primes; and that 3 feet, when multiplied by 3 inches, are assumed to give 3×3 superficial primes.

We are really assuming the law that, by multiplying the number of *feet* in one of the dimensions of a rectangle by the number of *inches* in the other dimension, we obtain the number of superficial primes in the area.

This we will proceed to prove.



Consider the rectangle *ABCD*.

Let *AB* represent a length of 3 ft., and let *BC* represent a length of 4 in.

Divide *AB* into three equal parts, so that each part will represent a foot.

Divide *BC* into four equal parts, so that each part will represent an inch.

Through the points of division draw straight lines parallel to *BC* and *AB* respectively.

We have thus divided the rectangle into several equal parts, each part representing a superficial prime.

And their number is 3×4; that is, (number of feet in *AB*) $\times$ (number of inches in *BC*).

Hence we arrive at the general result—

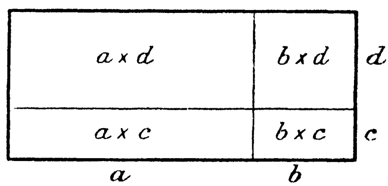
Linear feet × linear inches = superficial primes.

The scheme of work in the above example may be shortened thus:

ft.	in.	
3	7	
2	3	
7	2	
	10	9
8	0	9

Working.—7 in. × 2 ft. give 14 primes: put down 2 primes and carry 1 sq. ft. 3 ft. × 2 ft. give 6 sq. ft. and 1 to carry: put down 7 sq. ft. 7 in. × 3 in. give 21 sq. in.: put down 9 sq. in. and carry 1 prime. 3 ft. × 3 in. give 9 primes and 1 to carry: put down 10 primes. Add: 9 sq. in. 10 and 2 make 12 primes: put down 0 primes and carry 1 sq. ft. 1 and 7 make 8 sq. ft.

The plan of work here depends upon that property of a rectangle whose length measures $(a + b)$ units, and whose breadth measures $(c + d)$ units, viz. that its area is the sum of four rectangles measuring $a \times d$, $a \times c$, $b \times d$, $b \times c$ square units respectively.



We have found that, when dealing with a rectangle—

feet in length	× feet in breadth	=	square feet in area
feet ,,	× inches ,,	=	superficial primes in area
inches ,,	× inches ,,	=	square inches in area

or—

$$a \text{ ft.} \times b \text{ ft.} = (ab) \text{ sq. ft.}$$
$$a \text{ ft.} \times b \text{ in.} = (ab) \text{ superficial primes}$$
$$a \text{ in.} \times b \text{ in.} = (ab) \text{ sq. in.}$$

where a and b are the numbers which tell the measures of the dimensions of the rectangle.

Let us now introduce the twelfth part of an inch as an additional unit of length.

Then a rectangle measuring $\frac{1}{12}$ in. by $\frac{1}{12}$ in. will measure $\frac{1}{144}$ sq. in. in area . . . . § 8.

and a rectangle measuring 1 in. by $\frac{1}{12}$ in. will measure $\frac{1}{12}$ sq. in. in area . . . . § 8.

and a rectangle measuring 1 ft. by $\frac{1}{12}$ in. will measure 1 sq. in. in area . . . . . . § 8.

And we can prove as above that—

$$a\left(\tfrac{1}{12} \text{ in.}\right) \times b\left(\tfrac{1}{12} \text{ in.}\right) = ab\left(\tfrac{1}{144} \text{ sq. in.}\right)$$
$$a \text{ in.} \times b\left(\tfrac{1}{12} \text{ in.}\right) = ab\left(\tfrac{1}{12} \text{ sq. in.}\right)$$
$$a \text{ ft.} \times b\left(\tfrac{1}{12} \text{ in.}\right) = ab \text{ sq. in.}$$

where a and b are the numbers which tell the measures of the dimensions of the rectangle.

We have hitherto spoken of a plane rectangle measuring 1 ft. by 1 in. as a superficial prime. We shall now extend the application of the term *prime* to the twelfth part of any standard unit. And so we shall speak of the twelfth part of a linear foot (that is, an inch) as a linear prime, and of the twelfth part of a cubic foot as a cubic or solid prime.

Furthermore, the twelfth part of a prime, whether it be linear, superficial, or solid, we shall call a *second.*

And this duodecimal subdivision may be carried on to any extent.

Thus we have—

$$\begin{aligned} 1 \text{ standard unit} &= 12 \text{ primes (written } 12'), \\ 1 \text{ prime} &= 12 \text{ seconds (written } 12''), \\ 1 \text{ second} &= 12 \text{ thirds (written } 12'''), \\ 1 \text{ third} &= 12 \text{ fourths (written } 12''''), \\ &\text{and so on.} \end{aligned}$$

We may now express the above results as follows :—

When dealing with a rectangle—

$$\begin{aligned} a\,\textit{ft.} \times b\,\textit{ft.} &= ab\ \textit{sq. ft.} \\ a\,\textit{ft.} \times b\ \textit{linear primes} &= ab\ \textit{superficial primes} \\ a\,\textit{ft.} \times b\ \text{,,}\ \ \textit{seconds} &= ab\ \text{,,}\ \ \textit{seconds} \\ a\ \textit{linear primes} \times b\ \text{,,}\ \ \textit{primes} &= ab\ \text{,,}\ \ \text{,,} \\ a\ \text{,,}\ \ \text{,,} \times b\ \text{,,}\ \ \textit{seconds} &= ab\ \text{,,}\ \ \textit{thirds} \\ a\ \text{,,}\ \ \textit{seconds} \times b\ \text{,,}\ \ \textit{seconds} &= ab\ \text{,,}\ \ \textit{fourths} \end{aligned}$$

or—

$$\begin{aligned} \textbf{a ft.} \times \textbf{b ft.} &= \textbf{ab sq. ft.} \\ \textbf{a ft.} \times \mathbf{b'} &= (\mathbf{ab})' \\ \textbf{a ft.} \times \mathbf{b''} &= (\mathbf{ab})'' \\ \mathbf{a'} \times \mathbf{b'} &= (\mathbf{ab})'' \\ \mathbf{a'} \times \mathbf{b''} &= (\mathbf{ab})''' \\ \mathbf{a''} \times \mathbf{b''} &= (\mathbf{ab})'''' \end{aligned}$$

where the index shows the *order* of the unit.

We may here notice a relation which exists between the *orders* of the factors and the *order* of their product, and from the above results we may deduce the following rule :—

The order of the product is the sum of the orders of its factors.

ILLUSTRATIVE EXAMPLES.

14. *Example* 1.—Express 15 ft. $8\frac{1}{2}$ in. in duodecimals.

$$15 \text{ ft. } 8\tfrac{1}{2} \text{ in.} = \left(15 + \frac{8\frac{1}{2}}{12}\right) \text{ft.}$$

$= (15 + \frac{8}{12} + \frac{1}{24})$ ft.
$= (15 + \frac{8}{12} + \frac{6}{144})$ ft.
$= 15$ ft. $8' 6''$

Example 2.—Express 29 sq. ft. 76 sq. in. in duodecimals.

29 sq. ft. 76 sq. in. $= 29\frac{76}{144}$ sq. ft.
$= (29 + \frac{72}{144} + \frac{4}{144})$ sq. ft.
$= (29 + \frac{6}{12} + \frac{4}{144})$ sq. ft.
$= 29$ sq. ft. $6' 4''$

Example 3.—Express 105 cub. ft. $837\frac{1}{3}$ cub. in. in duodecimals.

105 cub. ft. $837\frac{1}{3}$ cub. in. $= 105\frac{837\frac{1}{3}}{1728}$ cub. ft.

$$= \left(105 + \frac{720}{1728} + \frac{117\frac{1}{3}}{1728}\right) \text{ cub. ft.}$$

$= (105 + \frac{5}{12} + \frac{108}{1728} + \frac{9}{1728} + \frac{1}{5184})$ cub. ft.
$= (105 + \frac{5}{12} + \frac{9}{144} + \frac{9}{1728} + \frac{4}{20736})$ cub. ft.
$= 105$ cub. ft. $5' 9'' 9''' 4^{iv}$

Example 4.—Find by duodecimals the area of a rectangle which measures 7 ft. 9 in. by 5 ft. 10 in.

Area of rectangle = 7 ft. 9 in. × 5 ft. 10 in. . . . § 8.

ft.	in.	
7	9	
5	10	
38	9	
6	5	6
45	2	6

∴ the area of rectangle = 45 sq. ft. 2 superficial primes 6 sq. in.

Example 5.—Find by duodecimals the area of a rectangle which measures 6 ft. 7 in. 4 twelfths of an inch by 3 ft. 8 in. 7 twelfths of an inch.

Area of rectangle = 6 ft. 7 in. 4 twelfths × 3 ft. 8 in. 7 twelfths . . § 8.
= 6 ft. 7 primes 4 seconds × 3 ft. 8 primes 7 seconds
= 6 ft. $7' 4''$ × 3 ft. $8' 7''$

6 ft.	$7'$	$4''$		
3 ft.	$8'$	$7''$		
19	$10'$	$0''$		
4	$4'$	$10''$	$8'''$	
	$3'$	$10''$	$3'''$	4^{iv}
24	$6'$	$8''$	$11'''$	4^{iv}

∴ area of rectangle = 24 sq. ft. 6 superf. primes 8 sq. in (or superf. seconds) 11 superf. thirds 4 superf. fourths

Examples—III.

Express the following lengths, areas, and volumes in duodecimals:—

1. 3 ft. 7 in.
2. 13 ft. $8\frac{1}{2}$ in.
3. 9 ft. $6\frac{3}{4}$ in.
4. 10 ft. $7\frac{1}{3}$ in.
5. 8 ft. $9\frac{1}{8}$ in.
6. 6 sq. ft. 24 sq. in.
7. 10 sq. ft. 52 sq. in.
8. 8 sq. ft. 103 sq. in.
9. 13 cub. ft. 326 cub. in.
10. 12 cub. ft. 731 cub. in.
11. 16 cub. ft. 963 cub. in.
12. 18 cub. ft. $1362\frac{1}{2}$ cub. in.

Find, by duodecimals, the areas of the rectangles having the following dimensions:—

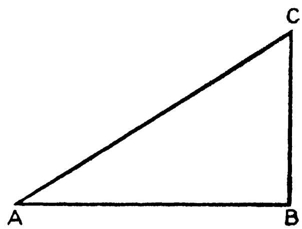
13. 3 ft. 7 in.; 2 ft.
14. 4 ft.; 3 ft. 6 in.
15. 7 ft. 9 in.; 5 ft. 4 in
16. 8 ft. 2 in.; 3 ft. 10 in.
17. 6 ft. 8 in.; 7 ft. 4 in.
18. 7 in. 9 twelfths; 6 in.
19. 8 in. 8 twelfths; 7 in. 5 twelfths.
20. 6 ft. 7 in. 10 twelfths; 5 ft. 8 in.
21. 10 ft. 6 in. 8 twelfths; 7 ft. 5 in. 4 twelfths.
22. 9 ft. 10 in. 7 twelfths; 4 ft. 6 in. 8 twelfths.

CHAPTER IV.

ON RIGHT-ANGLED TRIANGLES.

15. *A right-angled triangle* is a three-sided rectilineal figure which has a right angle. The side of a right-angled triangle which is opposite to the right angle is called the *hypotenuse.*

The sides of a right-angled triangle which contain the right angle are called the *base* and *perpendicular* respectively.

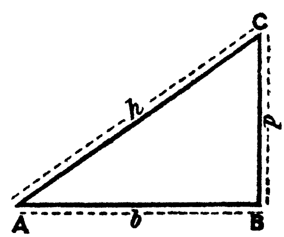


Thus, in the right-angled $\triangle ABC$, AC is the hypotenuse, AB is the base, and BC is the perpendicular.

The base and perpendicular of a right-angled triangle are often spoken of as "the sides which contain the right angle."

PROPOSITION II.

16. *To find the hypotenuse of a right-angled triangle, having given its base and perpendicular.*



Let ABC be a right-angled triangle. Let its base AB and its perpendicular BC measure b and p of the same linear unit respectively. It is required to find the hypotenuse in terms of b and p.

Now, since—

square on AC = square on AB + square on BC . Euc. I. 47.

and square on AB measures b^2 sq. units . . § 9.

also square on BC measures p^2 sq. units . . § 9.

$\therefore$ square on $AC = (b^2 + p^2)$ sq. units

$\therefore AC = \sqrt{b^2 + p^2}$ linear units . § 9.

Hence rule—

Add the square of the number of any linear unit in the base of a

right-angled triangle to the square of the number of the same linear unit in the perpendicular; then the square root of the sum will give the number of the same linear unit in the hypotenuse.

Or briefly—

$$\textbf{hypot. of right-angled triangle} = \sqrt{(\text{base})^2 + (\text{perpen.})^2}$$

$$\mathbf{h} = \sqrt{\mathbf{b}^2 + \mathbf{p}^2} \quad . \quad . \quad \text{(i.)}$$

Again, because—

$$(\text{hypot.})^2 = (\text{base})^2 + (\text{perpen.})^2$$

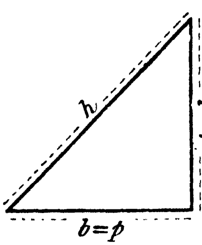
$$\therefore \begin{cases} (\text{base})^2 = (\text{hypot.})^2 - (\text{perpen.})^2 \\ (\text{perpen.})^2 = (\text{hypot.})^2 - (\text{base})^2 \end{cases}$$

$$\therefore \begin{cases} \textbf{base} = \sqrt{(\textbf{hypot.})^2 - (\textbf{perpen.})^2} \\ \textbf{perpen.} = \sqrt{(\textbf{hypot.})^2 - (\textbf{base})^2} \end{cases}$$

$$\therefore \begin{cases} \mathbf{b} = \sqrt{\mathbf{h}^2 - \mathbf{p}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)} \\ \mathbf{p} = \sqrt{\mathbf{h}^2 - \mathbf{b}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)} \end{cases}$$

Note.—It will often be found convenient to use the expression $\sqrt{(h-p)(h+p)}$ instead of $\sqrt{h^2-p^2}$, and the expression $\sqrt{(h-b)(h+b)}$ instead of $\sqrt{h^2-b^2}$, especially when large numbers are involved.

PARTICULAR CASES.



17. 1. Isosceles right-angled triangle.

Here base = perpendicular

Now, hypotenuse of any right-angled triangle

$$= \sqrt{(\text{base})^2 + (\text{perpen.})^2} \ . \ . \ . \ \S\ 16.$$

$\therefore$ hypotenuse of isosceles right-angled triangle

$$= \sqrt{2 \times (\text{base})^2} \text{ or } \sqrt{2 \times (\text{perpen.})^2}$$

$$h = \sqrt{2b^2} \text{ or } \sqrt{2p^2}$$

$$= b\sqrt{2} \text{ or } p\sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Therefore, by reversion—

$$\left.\begin{array}{r}\text{base or perpendicular of isos-} \\ \text{celes right-angled triangle}\end{array}\right\} = \frac{\text{hypot.}}{\sqrt{2}}$$

$$b = \frac{h}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$p = \frac{h}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

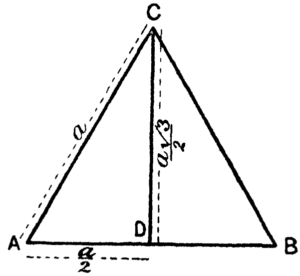
Note.—The diagonal of a square is the hypotenuse of an isosceles right-angled triangle.

2. Equilateral triangle.

Here the three sides are equal.

Let each side measure a of any linear unit. That is, $AB = BC = CA = a$ linear units.

Then, if CD be perpendicular to AB—



$$AD = \frac{a}{2} \text{ linear units}$$

Now, $CD^2 = AC^2 - AD^2$. § 16.

$$\therefore\ CD^2 = \left(a^2 - \frac{a^2}{4}\right) \text{ square units}$$

$$= \frac{3a^2}{4} \text{ square units}$$

$$\therefore\ CD = \frac{a\sqrt{3}}{2} \text{ linear units}$$

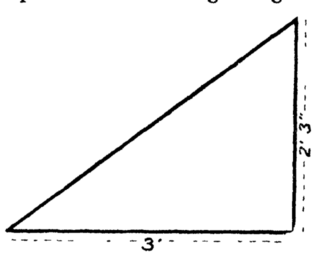
This important result may be stated thus :

The height of an equilateral triangle = side × $\frac{\sqrt{3}}{2}$ $\sqrt{3} = 1{\cdot}732$

$= \text{side} \times \frac{1{\cdot}732}{2}$

ILLUSTRATIVE EXAMPLES.

18. *Example* 1.—The base and perpendicular of a right-angled triangle measure 3 ft. and 2 ft. 3 in. respectively : find the hypotenuse.

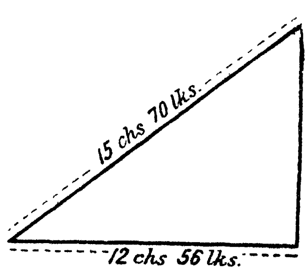


Hypotenuse $= \sqrt{b^2 + p^2}$ in. § 16.

where $b = 3 \times 12 = 36$,
and $p = (2 \times 12 + 3) = 27$;

$\therefore$ hypotenuse $= \sqrt{(36)^2 + (27)^2}$ in.
$= \sqrt{2025}$ in.
$= 45$ in.
$= 3$ ft. 9 in.

Example 2.—A plot of ground in the shape of a right-angled triangle measures 15 ch. 70 lks. along its hypotenuse, and 12 ch. 56 lks. along one of its other boundaries : find the length of its remaining boundary.

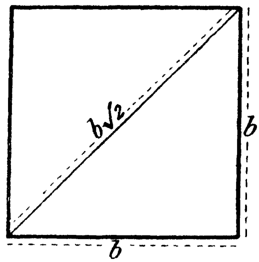


$\left.\begin{matrix}\text{Remaining}\\ \text{boundary}\end{matrix}\right\} = \sqrt{h^2 - b^2}$ lks. . § 16.

where $h = 1570$,
and $b = 1256$;

$\therefore$ Remaining boundary
$= \sqrt{(1570)^2 - (1256)^2}$ lks.
$= \sqrt{(1570 - 1256)(1570 + 1256)}$ lks.
$= \sqrt{887364}$ lks.
$= 942$ lks.
$= 9$ ch. 42 lks.

Example 3.—The side of a square measures 2 ft. 9 in. : find the length of its diagonal.



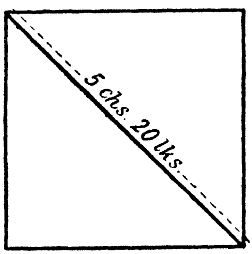
The diagonal of a square is the hypotenuse of an isosceles right-angled triangle, whose base and perpendicular are sides of the square. . . . . § 17.

$\therefore$ diagonal of square $= b\sqrt{2}$ in. § 17.

if $b =$ the number of inches in the side of the square $= (2 \times 12 + 9) = 33$.

$$\begin{aligned}\therefore \text{diagonal of square} &= 33 \times \sqrt{2} \text{ in.}\\ &= 33 \times 1{\cdot}41421 \ldots \text{ in.}\\ &= 46{\cdot}66 \ldots \text{ in.}\end{aligned}$$

Example 4.—The diagonal of a square measures 5 ch. 20 lks. : find the length of a side.



Side of square $= \dfrac{h}{\sqrt{2}}$ ch. . § 17.

where $h = 5{\cdot}2$;

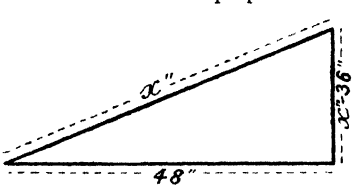
$$\begin{aligned}\therefore \text{side of square} &= \frac{5{\cdot}2}{\sqrt{2}} \text{ ch.}\\ &= \frac{5{\cdot}2 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \text{ ch.}\\ &= \frac{5{\cdot}2 \times \sqrt{2}}{2} \text{ ch.}\\ &= 2{\cdot}6 \times \sqrt{2} \text{ ch.}\\ &= 3{\cdot}6769 \ldots \text{ ch.}\\ &= 3 \text{ ch. } 67{\cdot}69 \ldots \text{ lks.}\end{aligned}$$

Example 5.—The base of a right-angled triangle is 48 in., and the difference between the hypotenuse and the perpendicular is 36 in. : find the hypotenuse and perpendicular.

Let x in. = the hypotenuse.

Then $(x - 36)$ in. = the perpendicular

But perpendicular $= \sqrt{h^2 - b^2}$ in. . . . . . . § 16.



where $h = x$,

$b = 48$;

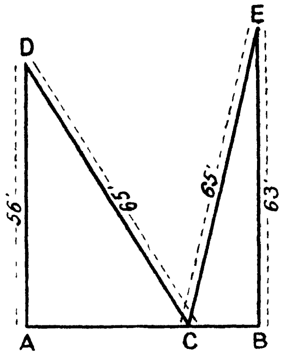
$$\begin{aligned}\therefore \text{perpendicular} &= \sqrt{x^2 - (48)^2} \text{ in.}\\ \therefore x - 36 &= \sqrt{x^2 - (48)^2}\\ \therefore x^2 - 72x + (36)^2 &= x^2 - (48)^2\\ \therefore 72x &= 2304 + 1296\\ &= 3600\\ \therefore x &= 50\end{aligned}$$

$\therefore$ hypotenuse = 50 in., and perpendicular = 14 in.

Example 6.—A ladder is placed so as to reach a window 63 ft. high. The ladder is then turned over to the opposite side of the street, and is found to reach a point 56 ft. high. If the ladder is 65 ft.

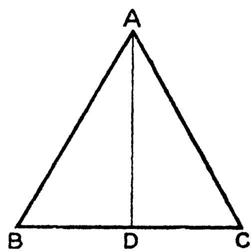
long, find the width of the street.

Let the line AB denote the width of the street, and let the point C indicate the foot of the ladder. Then we shall have the following measurements :—



$$\begin{aligned} CE &= CD = 65 \text{ ft.} \\ BE &= 63 \text{ ft.} \\ AD &= 56 \text{ ft.} \\ \text{hence } AC &= \sqrt{(65)^2-(56)^2} \text{ ft. § 16.} \\ &= 33 \text{ ft.} \\ \text{and } CB &= \sqrt{(65)^2-(63)^2} \text{ ft. § 16.} \\ &= 16 \text{ ft.} \\ \text{But width of street} &= AC + CB \\ \therefore \text{ width of street} &= (33 + 16) \text{ ft.} \\ &= 49 \text{ ft.} \end{aligned}$$

Example 7.—The cost of fencing an enclosure, in the form of an equilateral triangle, at 3 annas a foot, amounts to Rs.56 4 annas : find the distance from an angular point to the middle point of the opposite side.



$$\begin{aligned} \text{Perimeter of triangle} &= \frac{\text{Rs.56 4 annas}}{\text{3 annas}} \text{ ft.} \\ &= \tfrac{900}{3} \text{ ft.} \\ &= 300 \text{ ft.} \\ \therefore \text{ each side of triangle} &= 100 \text{ ft.} \end{aligned}$$

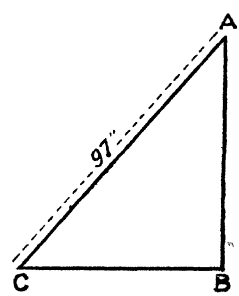
But if AB, a side of the equilateral $\triangle$ ABC, measure 100 ft., then AD, the perpendicular from A on BC, that is, the distance from any angular point to the middle point of the opposite side, will measure—

$$\begin{aligned} \frac{100\sqrt{3}}{2} \text{ ft.} &= 50 \times 1{\cdot}732 \ldots \text{ ft.} \quad \ldots \text{ § 17.} \\ &= 86{\cdot}6 \ldots \text{ ft.} \end{aligned}$$

* *Example* 8.—The perimeter of a right-angled triangle measures 234 in., and the hypotenuse measures 97 in. : find the other two sides.

Let ABC be the right-angled triangle. Then AC will measure 97 in.

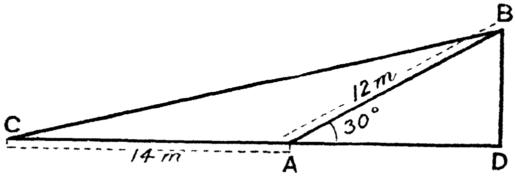
Let AB and BC measure x in. and y in. respectively.



$$\begin{aligned} \text{Now, } (x+y)^2+(x-y)^2 &= 2(x^2+y^2) \\ &= 2 \times (97)^2 \quad . \quad . \text{ § 16.} \\ \text{But } x+y &= 234 - 97 = 137 \\ \therefore x - y &= \sqrt{2 \times (97)^2 - (137)^2} \\ &= \sqrt{18818 - 18769} \\ &= \sqrt{49} \\ &= 7 \\ \text{But } x + y &= 137 \\ \therefore x &= 72 \\ \text{and } y &= 65 \end{aligned}$$

Thus the other two sides of the triangle measure 72 in. and 65 in. respectively.

Example 9.—Two roads diverge from a point at an angle of 150° to each other. Two persons, one on each road, start from the point at the rate of 3 and $3\frac{1}{2}$ mi. an hour respectively, and travel for 4 hours along their respective roads. What will be the direct distance between them at the end of the fourth hour?



If AB and AC denote the distances traversed by the two persons respectively, then CB will denote the direct distance between them at the end of the fourth hour.

$$\text{Since } \angle CAB = 150^\circ$$
$$\therefore \angle DAB = 30^\circ$$
$$\text{and } \angle DBA = 60^\circ$$
$$\therefore AD = \frac{AB\sqrt{3}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 17.$$

But AB measures 3×4 mi. $= 12$ mi.

$$\therefore AD \text{ measures } 6\sqrt{3} \text{ mi.}$$
$$\therefore CD = CA + AD = (14 + 6\sqrt{3}) \text{ mi.}$$

Again—

$$BD = \tfrac{1}{2} . AB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 17.$$
$$= 6 \text{ mi.}$$
$$\text{and, } CB = \sqrt{CD^2 + DB^2} \quad . \quad . \quad . \quad . \quad \S\ 16.$$
$$\therefore CB = \sqrt{(14 + 6\sqrt{3})^2 + (6)^2} \text{ mi.}$$
$$= \sqrt{340 + 168\sqrt{3}} \text{ mi.}$$
$$= \sqrt{630\cdot9845 \ldots} \text{ mi.}$$
$$= 25\cdot11 \ldots \text{ mi.}$$

Example 10.—In a right-angled triangle the perpendicular is 6 in. shorter than $\frac{11}{12}$ of the base, and the hypotenuse is 3 in. shorter than $\frac{4}{3}$ of the base: find the base.

Let the base measure x in.

Then the perpendicular will measure $(\frac{11}{12}x - 6)$ in., and the hypotenuse will measure $(\frac{4}{3}x - 3)$ in.

$$\text{But (hypotenuse)}^2 = (\text{base})^2 + (\text{perpendicular})^2. \quad . \quad . \quad \S\ 16.$$
$$\therefore (\tfrac{4}{3}x - 3)^2 = x^2 + (\tfrac{11}{12}x - 6)^2$$

Solving this equation, we find—

$$x = 36$$

Hence the base measures 36 in.

Examples—IV. A.

Find the hypotenuse of each of the following right-angled triangles, whose sides containing the right angle are respectively—

1. 30 in.; 72 in.

2. 75 yds.; 100 yds.

3. 176 yds.; 15 yds. 1 ft.

4. 9 yds. 2 ft. 2 in.; 3 yds. 1 ft. (give the result in yards, feet, and inches).

5. 1 mi. 5 fur. 12 po.; 4 fur. 5 po. (give the result in miles, furlongs, and poles).

6. 7 ch. 14 lks.; 9 ch. 52 lks. (give the result in chains and links).

7. 16 ch. 50 lks.; 22 ch. (give the result in chains and links).

8. 2740 lks., 2877 lks. (give the result in chains).

Find the remaining side of each of the following right-angled triangles, of which the hypotenuse and one side are respectively—

9. 162 lks.; 136 lks.

10. 6·5 in.; 5·6 in.

11. 2 ft. 1 in.; 2 ft.

12. 9 mi. 2 fur.; 8 mi. 6 fur.

13. 7 ch. 80 lks.; 3 ch. 96 lks.

14. 8 mi. 6 fur. 36 po.; 3 mi. 1 fur. 36 po.

15. One side of a right-angled triangle is 126 yds.; the difference between the hypotenuse and the other side is 42 yds.: find the hypotenuse and the other side.

16. The side of a square is 1 ft. 6 in.: find the distance of the central point of the square from each corner.

17. A ladder, 15 ft. long, is standing upright against a wall: how far must the lower end of the ladder be drawn away from the wall so that the upper end may be lowered 3 ft.?

18. The sum of the hypotenuse and one side of a right-angled triangle is 153 in., and the other side is 51 in.: find the hypotenuse.

19. The dimensions of a rectangle are 2 ft. and 3 ft. 9 in. respectively: find the diagonal.

20. Find the diagonal of a square whose side measures 13 in.

21. Each of the sides of an isosceles triangle measures 4 ft. 2 in., and the base measures 6 ft. 8 in.: find the perpendicular from the vertex upon the base.

22. Find the height of a window above the ground if it is just reached by a ladder 9 ft. 2 in. long, the foot of the ladder being 5 ft. 6 in. from the side of the house.

23. The hypotenuse of a right-angled triangle is 5 ch., and its perpendicular is twice its base: find its perpendicular correct to two places of decimals.

24. Find the cost, to the nearest anna, of putting a fence round an enclosure in the shape of a right-angled isosceles triangle, whose hypotenuse measures 100 yds., at Rs.3 a yard.

25. If a man can traverse a square from corner to corner in 4 min., walking 3 mi. an hour, find to the nearest foot the perimeter of the square.

26. A ladder is placed so as to reach a window 36 ft. high. The ladder is then turned over to the opposite side of the street, and is found to reach a point 24 ft. high. If the ladder is 40 ft. long, find the width of the street.

27. A town, *A*, is 132 mi. due north of another town, *B*, and 204 mi. due west of a third town, *C*: how far is *B* from *C*?

28. A piece of rope, 46 ft. 5 in. long, reaches from the top of a flagstaff to a point on the ground 13 ft. 9 in. from the foot of the flagstaff: find the height of the flagstaff.

29. A moat lies in front of a tower 100 ft. high: find the width of the moat if a line, 118 ft. long, will reach from the top of the tower to the opposite bank.

30. The hypotenuse of a right-angled triangle is 18 ch.: find the lengths of the two sides, one of which is three-fourths the other.

Examples—IV. B.

Find the hypotenuse of each of the following right-angled triangles, whose sides containing the right angle are respectively—

31. 24 rasi; 7 rasi.
32. 13 lathas; 84 lathas.
33. 85 girahs; 132 girahs.
34. 60 haths; 221 haths.
35. 76 guz; 357 guz.
36. 145 rasi; 408 rasi.

Find the remaining side of each of the following right-angled triangles, of which the hypotenuse and one side are respectively—

37. 41 lathas; 40 lathas.
38. 53 rasi; 45 rasi.
39. 109 girahs; 91 girahs.
40. 137 haths; 88 haths.
41. 205 guz; 187 guz.
42. 409 rasi; 391 rasi.

Examination Questions—IV.

A. *Allahabad University: Matriculation.*

1. In a right-angled triangle the difference of the sides is 21 ft., and the hypotenuse is 39 ft.: find both the sides.

2. A ladder, 24 ft. long, stands upright against a wall: how far must the bottom of the ladder be pulled out so as to lower the top 3 ft.?

3. A tower, which stands in a horizontal plane, subtends a certain angle at a point 160 ft. from the foot of the tower. On advancing 100 ft. towards it, the tower is found to subtend an angle twice as great as before. What is the height of the tower?

B. *Punjab University: Matriculation.*

4. Two roads diverge from a point at an angle of 120° to each other. Two persons, one on each road, start from the point at the rate of 4 and 5 miles per hour respectively. What will be the direct distance between the persons after they shall have travelled for 6 hours on their respective roads?

5. Given the perimeter of a right-angled isosceles triangle $= \sqrt{2} + 1$, find the hypotenuse.

6. Given the sum or difference of the hypotenuse and one side, and also the remaining side, find the hypotenuse.

7. ABC is a right-angled triangle, right angled at B; D is a point in AB; $BD = BC = 33$ ft. $= \frac{1}{2}(AD + AC)$: find AB.

C. *Punjab University: Middle School.*

8. Find to five decimal places the diagonal of a square of which the side is one mile. Find in acres the area of the square.

9. The hypotenuse of a right-angled triangle is 123 ft., and one side is 9 yds.: find the other side.

10. The side of a square is 6 yds. : find the radius of the circle described round the square.

D. *Calcutta University: Matriculation.*

11. A man on one side of a brook finds that he can just rest a ladder 20 ft. long against the branch of a tree vertically over the other bank; the branch is 12 ft. above the ground: how wide is the brook?

E. *European Schools: Final. United Provinces.*

12. Prove that any triangle that has sides in the ratio $3 : 4 : 5$ must be right-angled.

13. The breadth of the bottom of a ditch is to be 16 ft., and the depth 9 ft., and the inclinations of the sides to the top 30° and 45°: what must be the breadth of the excavation at the top?

F. *Madras Technical: Elementary.*

14. The span of a roof is 21 ft., and the rise 7 ft. : find the slope length of each side.

G. *Madras Technical: Intermediate.*

15. The radius of a circle is 1 ft. : find the area of a square inscribed in the circle.

H. *Roorkee Engineer: Entrance.*

16. A square field contains 31 ac. 0 ro. 10·25 sq. po. : find the length of its diagonal.

17. From a point within a rectangle, lines measuring 16 and 20 in. are drawn to opposite angles; a third line measuring 12 in. is also drawn to one of the other angles: find the length of the line drawn from the point to the remaining angle.

I. *Roorkee Upper Subordinate: Entrance.*

18. One side of a right-angled triangle is 588 ft.; the sum of the hypotenuse and other side is 882 ft. : find the hypotenuse and other side.

19. In the middle of a pond 10 ft. by 10 ft. grew a reed which raised its head 1 ft. above the surface of the water. A person standing on the brink, at a middle point of one of the sides, could just draw the top of the reed to the edge of the bank. How deep was the water?

20. One side of a right-angled triangle is 3925 ft.; the difference between the hypotenuse and the other side is 625 ft. : find the hypotenuse and the other side.

J. *Roorkee Upper Subordinate: Monthly.*

21. A ladder, 25 ft. long, is placed against a wall with its foot 7 ft. from the wall: how far should the foot be drawn out so that the top of the ladder may come down by half the distance that the foot is drawn out?

22. The diagonal of a square court is 300 ft. : find its area in square yards.

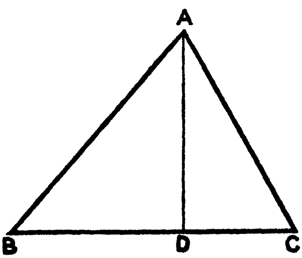
23. Find the area of a square whose perimeter is 3000 ft.

24. What is the length of the diagonal of the greatest square that can be cut out of a right-angled triangle, each of whose legs is 40 ft.?

25. What would be the cost of thatching a hipped roof of the following dimensions at Rs.10 per 100 superficial feet, length and breadth of eaves 108 ft. and 36 ft. respectively, and slope of roof 45°? $\sqrt{2} = 1{\cdot}41421$.

CHAPTER V.

ON ANY TRIANGLES.



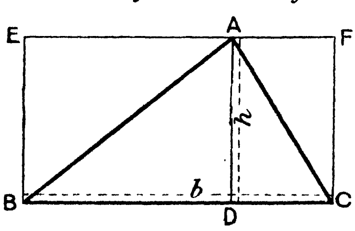
19. A *triangle* is a figure bounded by three straight lines.

The side of a triangle on which it may be supposed to stand is called its *base.* The *height* of a triangle is the perpendicular drawn to the base from the opposite angular point.

Thus, in the $\triangle ABC$, BC may be regarded as the base, and AD as the height.

PROPOSITION III.

20. *To find the area of a triangle, having given its base and height.*



Let ABC be a triangle.

Let its base BC and its height AD measure b and h of the same linear unit respectively. It is required to find the area of the triangle in terms of b and h.

Construct a rectangle EC on the same base BC, and of the same height AD.

Now, since—

$$\text{area of } \triangle ABC = \tfrac{1}{2} \text{ area of rectangle } EC \quad . \quad \text{Euc. I. 41.}$$
$$\therefore \text{area of } \triangle ABC = \tfrac{1}{2} \times BC \times CF \; . \; . \; . \; . \; . \; . \; . \; . \; \S\, 8.$$
$$= \tfrac{1}{2} \times BC \times AD$$
$$= \tfrac{1}{2} \times b \times h \text{ square units}$$

Hence rule—

Multiply the number of any linear unit in the base of a triangle

by the number of the same linear unit in the height, then half the product will give the number of the corresponding square unit in the area.

Or briefly—

$$\text{The area of a triangle} = \tfrac{1}{2}\text{ base} \times \text{height}$$

$$A = \tfrac{1}{2}bh \quad \ldots\ldots \quad \text{(i.)}$$

$$\therefore \text{ the height of a triangle} = \frac{2 \times \text{area}}{\text{base}}$$

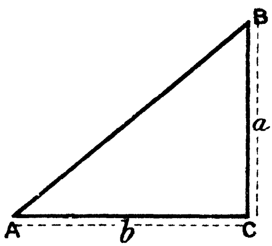
$$h = \frac{2A}{b} \quad \ldots\ldots \quad \text{(ii.)}$$

$$\text{and the base of a triangle} = \frac{2 \times \text{area}}{\text{height}}$$

$$b = \frac{2A}{h} \quad \ldots\ldots \quad \text{(iii.)}$$

PARTICULAR CASES.

21. 1. **Right-angled triangle.**



Let ABC be a right-angled triangle having a right angle at C.

Then, if AC be regarded as the base of the triangle, BC will evidently be its height.

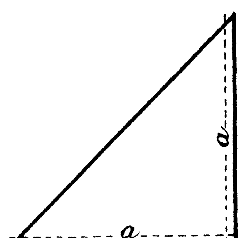
Now, area of any triangle $= \frac{1}{2}$ base $\times$ height . § 20.

$\therefore$ area of triangle $ABC = \frac{1}{2} \times AC \times BC$

$\therefore$ area of any right-angled triangle $= \frac{1}{2} \times$ (product of sides containing the right angle)

$= \frac{1}{2}ab$ square units

2. **Isosceles right-angled triangle.**



Here the sides containing the right angle are equal to one another.

That is, $a = b$

Hence—

Area of any isosceles right-angled triangle

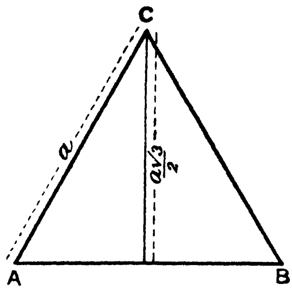
$= \frac{1}{2} \times$ (square of one of the sides containing the right angle)

$= \frac{1}{2}a^2$ square units

3. **Equilateral triangle.**

Let ABC be an equilateral triangle.

Let each side of the triangle measure a of any unit of length.



Then we know by § 17 that the height of the triangle will measure $\frac{a\sqrt{3}}{2}$ of the same unit of length.

Now, area of any triangle $= \frac{1}{2}$ base × height § 20

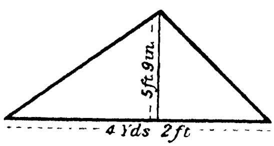
∴ area of equilateral triangle ABC

$$= \tfrac{1}{2} \times a \times \frac{a\sqrt{3}}{2} \text{ square units}$$

$$= \frac{\sqrt{3}}{4}a^2 \text{ square units}$$

ILLUSTRATIVE EXAMPLES.

22. *Example* 1.—The base of a triangle is 4 yds. 2 ft., and its height is 5 ft. 9 in.: find its area in square yards, etc.



Area of triangle $= \frac{1}{2}bh$ sq. ft. § 20.

where $b = (4 \times 3 + 2) = 14$,

and $h = 5\frac{3}{4}$;

∴ area of triangle $= \frac{1}{2} \times 14 \times 5\frac{3}{4}$ sq. ft.

$= 40\frac{1}{4}$ sq. ft.

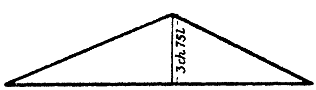
$=$ 4 sq. yds. 4 sq. ft. 36 sq. in

Example 2.—The area of a triangle is 3 ac., and its height is 3 ch. 75 lks.; find its base.

Base of triangle $= \frac{2 \times A}{h}$ ch. . . . . § 20.

where $A = 3 \times 10 = 30$,

and $h = 3{\cdot}75$;

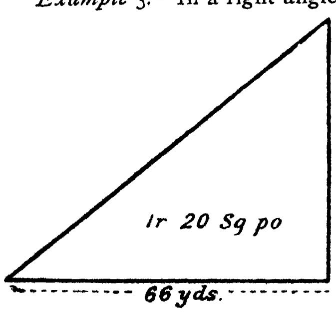


∴ base of triangle $= \frac{2 \times 30}{3{\cdot}75}$ ch.

$= \frac{60}{3{\cdot}75}$ ch.

$= 16$ ch.

Example 3.—In a right-angled triangle the area is 1 ro. 20 sq. po., and one of the sides containing the right-angle is 66 yds.: find the other side.



Area of right-angled triangle $= \frac{1}{2}ab$ sq. yds. § 21.

where $b = 66$;

$$\therefore a = \frac{2 \times \text{area}}{b} = \frac{2A}{66}$$

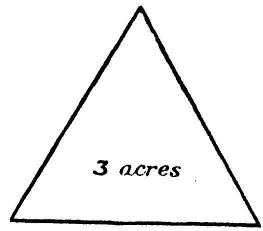
where $A = (1 \times 40 + 20) \times 30\frac{1}{4}$

$= 60 \times 30\frac{1}{4}$

∴ the other side $= \frac{2 \times 60 \times 30\frac{1}{4}}{66}$ yds.

$= 55$ yds.

Example 4.—Find the side of an equilateral triangle whose area measures 3 ac.



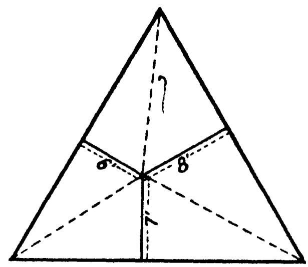
$$\left.\begin{array}{l}\text{Area of equilateral}\\ \quad\text{triangle}\end{array}\right\} = \frac{\sqrt{3}}{4}a^2 \text{ sq. ch.} \quad . \quad \S\ 21.$$

$$\therefore a^2 = \frac{4 \times \text{area}}{\sqrt{3}} = \frac{4A}{\sqrt{3}}$$

where $A = 3 \times 10$;

$$\left.\begin{array}{l}\therefore \text{ the side of the}\\ \text{equilateral triangle}\end{array}\right\} = \sqrt{\left(\frac{4 \times 3 \times 10}{\sqrt{3}}\right)} \text{ ch.}$$
$$= 8{\cdot}32 \ldots \text{ ch.}$$

Example 5.—From a point within an equilateral triangle perpendiculars are drawn to the three sides, and are 6, 7, and 8 ft. respectively: find the area of the triangle.



Let each side of the triangle measure a feet.

Then—

$$\left.\begin{array}{l}\text{Area of}\\ \text{triangle}\end{array}\right\} = (\tfrac{1}{2} . a . 6 + \tfrac{1}{2} . a . 7 + \tfrac{1}{2} . a . 8) \text{ sq. ft.} \quad \S\ 20.$$
$$= \frac{21a}{2} \text{ sq. ft.}$$

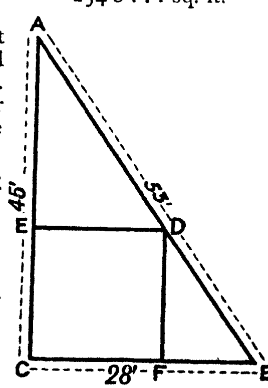
But area of equilateral triangle of side a ft. $= \frac{a^2\sqrt{3}}{4}$ sq. ft. . . § 21.

$$\therefore \frac{a^2\sqrt{3}}{4} = \frac{21a}{2}$$

$$\therefore a = \frac{42}{\sqrt{3}} = 14\sqrt{3}$$

$$\text{Hence area of triangle} = \frac{(14\sqrt{3})^2 . \sqrt{3}}{4} \text{ sq. ft.}$$
$$= 254{\cdot}6 \ldots \text{ sq. ft.}$$

* *Example* 6.—Find the side of the greatest square that can be cut out of a right-angled triangle whose sides measure 28, 45, 53 ft. respectively, in such a way that an angular point of the square lies on the hypotenuse of the triangle.



Let each side of the square measure x ft.

Now, if ABC be the triangle, and $DFCE$ the square,

	AE	will measure	$(45 - x)$ ft.
	BF	,, ,,	$(28 - x)$ ft.
Hence	$\triangle AED$	,, ,,	$\frac{1}{2} . (45 - x)x$ sq. ft. § 20.
and	$\triangle BFD$	,, ,,	$\frac{1}{2} . (28 - x)x$ sq. ft. § 20.

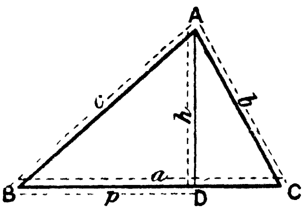
Thus we have—

$$\tfrac{1}{2}.(45 - x)x + \tfrac{1}{2}.(28 - x)x + x^2 = \tfrac{1}{2}.28.45$$
$$\therefore 73x = 28 \times 45$$
$$x = 17\tfrac{19}{73}$$

Hence side of square measures $17\frac{19}{73}$ ft.

PROPOSITION IV.

23. *To find the area of a triangle, having given its three sides.*



Let ABC be a triangle.

From A draw AD perpendicular to BC.

Let BC, CA, AB, AD, BD measure a, b, c, h, p of the same linear unit respectively. It is required to find the area of the triangle in terms of a, b, c.

Now, since—

area of any triangle $= \frac{1}{2} \times$ base $\times$ height . . § 20.

$\therefore$ area of $\triangle ABC = \frac{1}{2} \times BC \times AD$

$= \frac{1}{2} \times a \times h$ sq. units

It is now necessary to express h in terms of a, b, and c. To do this we shall find it convenient first to determine p.

Now, $AD^2 = AB^2 - BD^2$ . . . . § 16.

that is, $h^2 = c^2 - p^2$

Also $AD^2 = AC^2 - CD^2$ . . . . § 16.

that is, $h^2 = b^2 - (a - p)^2$ $(\because CD = BC - BD)$

$$\therefore c^2 - p^2 = b^2 - (a - p)^2$$
$$= b^2 - a^2 + 2ap - p^2$$
$$\therefore 2ap = c^2 + a^2 - b^2$$
$$\therefore p = \frac{c^2 + a^2 - b^2}{2a}$$

Again, $\because h^2 = c^2 - p^2$

$$\therefore h^2 = c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$$
$$= \left(c - \frac{c^2 + a^2 - b^2}{2a}\right)\left(c + \frac{c^2 + a^2 - b^2}{2a}\right)$$
$$= \frac{b^2 - (c^2 + a^2 - 2ca)}{2a} \times \frac{(c^2 + a^2 + 2ca) - b^2}{2a}$$
$$= \frac{b^2 - (c - a)^2}{2a} \cdot \frac{(c + a)^2 - b^2}{2a}$$
$$= \frac{(b + c - a)(b - c + a)(c + a - b)(c + a + b)}{4a^2}$$

Now put $s = \dfrac{a+b+c}{2}$

so that $s - a = \dfrac{a+b+c}{2} - a = \dfrac{b+c-a}{2}$

$$s - b = \frac{a+b+c}{2} - b = \frac{c+a-b}{2}$$

$$s - c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$$

and $\because h^2 = \dfrac{b+c-a}{2} \cdot \dfrac{a+b-c}{2} \cdot \dfrac{c+a-b}{2} \cdot \dfrac{a+b+c}{2} \times \dfrac{4}{a^2}$

$$\therefore h^2 = (s-a)(s-c)(s-b)s \times \frac{4}{a^2}$$

$$\therefore \tfrac{1}{2}h = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{a}$$

$\therefore$ area of triangle $= \frac{1}{2}ha$ sq. units $= \sqrt{s(s-a)(s-b)(s-c)}$ sq. units

Hence rule—

Subtract each side of a triangle separately from the semi-perimeter; multiply together the numbers of the same linear unit in each of the three remainders thus obtained and in the semi-perimeter; then the square root of this continued product will give the number of the corresponding square unit in the area.

Or briefly—

Area of triangle

$$= \sqrt{\left\{ \frac{\text{sum of sides}}{2} \times \left(\frac{\text{sum of sides}}{2} - \text{first side} \right) \times \left(\frac{\text{sum of sides}}{2} - \text{second side} \right) \times \left(\frac{\text{sum of sides}}{2} - \text{third side} \right) \right\}}$$

$$\mathbf{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

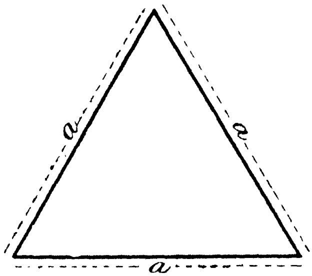
Note.—When the expression $s(s-a)(s-b)(s-c)$ is a perfect square, it is possible to write down its square root from inspection, by resolving it into factors.

Thus—

$$\begin{aligned} \sqrt{54(54-27)(54-36)(54-45)} &= \sqrt{54 \times 27 \times 18 \times 9} \\ &= \sqrt{9 \times 3 \times 2 \times 9 \times 3 \times 9 \times 2 \times 9} \\ &= 9 \times 9 \times 3 \times 2 \\ &= 486 \end{aligned}$$

PARTICULAR CASES.

24. 1. **Equilateral triangle.**



Here the three sides are equal.

That is, $a = b = c$

$$\therefore s = \frac{3a}{2}$$

Hence area of equilateral triangle

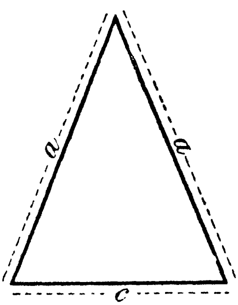
$$= \sqrt{\left\{\frac{3a}{2}\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a\right)\right\}} \text{ sq. units}$$

$$= \sqrt{\frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2}} \text{ sq. units}$$

$$= \frac{\sqrt{3}}{4}a^2 \text{ sq. units}$$

This result has been previously obtained in § 21.

2. **Isosceles triangle.**



Here two sides are equal.

That is, $a = b$

$$\therefore s = \frac{c + 2a}{2}$$

Hence area of isosceles triangle

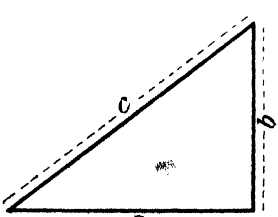
$$= \sqrt{\frac{c + 2a}{2}\left(\frac{c + 2a}{2} - a\right)\left(\frac{c + 2a}{2} - a\right)\left(\frac{c + 2a}{2} - c\right)} \text{ sq. units}$$

$$= \sqrt{\frac{c + 2a}{2} \cdot \frac{c}{2} \cdot \frac{c}{2} \cdot \frac{2a - c}{2}} \text{ sq. units}$$

$$= \frac{c}{4}\sqrt{4a^2 - c^2} \text{ sq. units}$$

3. **Right-angled triangle.**

Here $c = \sqrt{a^2 + b^2}$, if c is the measure of the hypotenuse. . . . . § 16.



$$\therefore s = \frac{a + b + \sqrt{a^2 + b^2}}{2}$$

and $$s - a = \frac{\sqrt{a^2 + b^2} + b - a}{2}$$

$$s - b = \frac{\sqrt{a^2 + b^2} + a - b}{2}$$

$$s - c = \frac{a + b - \sqrt{a^2 + b^2}}{2}$$

$\therefore$ area of right-angled triangle

$$= \sqrt{\left\{\frac{(a+b)+\sqrt{a^2+b^2}}{2} \times \frac{(a+b)-\sqrt{a^2+b^2}}{2}\right.}$$

$$\sqrt{\left.\times \frac{\sqrt{a^2+b^2}+(b-a)}{2} \times \frac{\sqrt{a^2+b^2}-(b-a)}{2}\right\}} \text{ sq. units}$$

$$= \sqrt{\left\{\frac{2ab}{4} \cdot \frac{2ab}{4}\right\}} \text{ sq. units}$$

$$= \frac{ab}{2} \text{ sq. units}$$

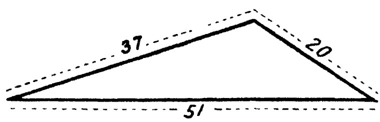
This result has been previously obtained in § 21.

ILLUSTRATIVE EXAMPLES.

25. *Example* 1.—Find the area of a triangle whose sides are 51, 37, and 20 yds. respectively.

Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ sq. yds. . § 23.

where $a = 51$, $b = 37$, $c = 20$;



$$\therefore s = \frac{51+37+20}{2} = 54$$

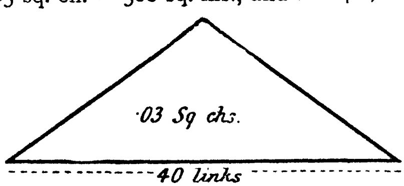
and $s - a = 54 - 51 = 3$
$s - b = 54 - 37 = 17$
$s - c = 54 - 20 = 34$

$\therefore$ area of triangle $= \sqrt{54 \cdot 3 \cdot 17 \cdot 34}$ sq. yds.
$= \sqrt{2^2 \times 3^2 \times 3^2 \times 17^2}$ sq. yds.
$= 2 \times 3 \times 3 \times 17$ sq. yds.
$= 306$ sq. yds.

Example 2.—Find each side of an isosceles triangle whose area measures 0·03 sq. ch., and whose base measures 40 lks.

Area of isosceles triangle $= \frac{c}{4}\sqrt{4a^2 - c^2}$ sq. lks. . § 24.

But area of triangle = 0·03 sq. ch. = 300 sq. lks., and $c = 40$;



$\therefore \frac{40}{4}\sqrt{4a^2-(40)^2} = 300$
$\therefore \sqrt{4a^2 - 1600} = 30$
$\therefore 4a^2 = 900 + 1600$
$= 2500$
$\therefore 2a = 50$
$\therefore a = 25$

Hence each side measures 25 lks.

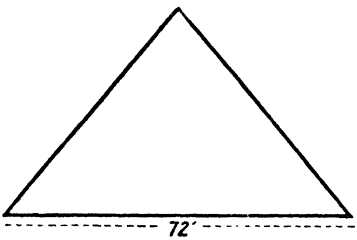
Example 3.—A lawn is in the form of an isosceles triangle. The cost of turfing it came to Rs.600 at 6 annas per square foot. If the base be 72 ft. long, find the length of each of the sides.

Area of isosceles triangle $= \frac{c}{4}\sqrt{4a^2 - c^2}$ sq. ft. . . . § 24.

where a = the number of feet in each side,
and $c = 72$;

$$\therefore \text{ area of isosceles triangle } = \tfrac{72}{4}\sqrt{4a^2 - (72)^2} \text{ sq. ft.}$$

$$\text{Now area of lawn } = \left(\frac{\text{total cost of turfing}}{\text{cost of turfing a square foot}}\right) \text{ sq. ft.}$$
$$= \frac{600 \times 16}{6} \text{ sq. ft.}$$
$$= 1600 \text{ sq. ft.}$$

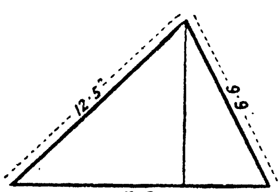


$$\therefore \tfrac{72}{4}\sqrt{4a^2 - (72)^2} = 1600$$
$$\therefore \sqrt{4a^2 - 5184} = \tfrac{1600}{18} = \tfrac{800}{9}$$
$$\therefore 4a^2 - 5184 = \tfrac{640000}{81}$$
$$\therefore 4a^2 = \tfrac{640000}{81} + 5184$$
$$= \tfrac{1059904}{81}$$
$$\therefore 2a = \sqrt{\tfrac{1059904}{81}}$$
$$= \tfrac{1029{\cdot}51+}{9}$$
$$\therefore a = 57{\cdot}19 +$$

Hence each side of the lawn measures $57{\cdot}19+$ ft.

Example 4.—The sides of a triangle are 13·6, 12·5, 9·9 in. respectively: find the distance of the longest side from the opposite vertex.

$$\text{Area of triangle } = \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. in.} \quad . \ \S\ 23.$$



where $a = 13{\cdot}6$,
$b = 12{\cdot}5$,
$c = 9{\cdot}9$;

$$\therefore s = \frac{13{\cdot}6 + 12{\cdot}5 + 9{\cdot}9}{2}$$
$$= 18$$

and
$$s - a = 18 - 13{\cdot}6 = 4{\cdot}4$$
$$s - b = 18 - 12{\cdot}5 = 5{\cdot}5$$
$$s - c = 18 - 9{\cdot}9 = 8{\cdot}1$$

$$\therefore \text{ area of triangle } = \sqrt{18 \times 4{\cdot}4 \times 5{\cdot}5 \times 8{\cdot}1} \text{ sq. in.}$$
$$= \sqrt{3528{\cdot}36} \text{ sq. in.}$$
$$= 59{\cdot}4 \text{ sq. in.}$$

$$\left.\begin{array}{r}\text{Now, } \tfrac{1}{2} \times \text{ required dis-} \\ \text{tance} \times \text{ longest side}\end{array}\right\} = \text{area of triangle} \quad . \quad . \quad . \quad . \ \S\ 20.$$

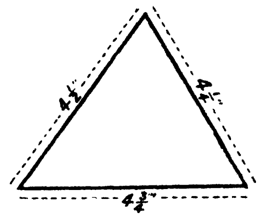
$$\therefore \tfrac{1}{2} \times \text{ required distance } = \frac{\text{area of triangle}}{\text{longest side}}$$
$$\therefore \tfrac{1}{2} \times \quad \text{,,} \quad \text{,,} \quad = \frac{59{\cdot}4}{13{\cdot}6} \text{ in.}$$
$$\therefore \quad \text{,,} \quad \text{,,} \quad = \frac{59{\cdot}4}{6{\cdot}8} \text{ in.}$$
$$= 8\tfrac{25}{34} \text{ in.}$$

Example 5.—The three sides of a triangle are $4\frac{1}{4}$, $4\frac{1}{2}$, and $4\frac{3}{4}$ in. respectively. If the first side is given as $3\frac{1}{4}$ in. instead of $4\frac{1}{4}$ in. by

mistake, what will be the extent of the error in computing the area of the triangle?

$$\text{True area} = \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. in.} \quad . \quad . \ \S\ 23.$$

where $a = 4{\cdot}25$,
$b = 4{\cdot}5$,
$c = 4{\cdot}75$;



$$\therefore s = \frac{4{\cdot}25 + 4{\cdot}5 + 4{\cdot}75}{2}$$
$$= \frac{13{\cdot}5}{2}$$
$$= 6{\cdot}75$$

and $s - a = 6{\cdot}75 - 4{\cdot}25 = 2{\cdot}5$
$s - b = 6{\cdot}75 - 4{\cdot}5 = 2{\cdot}25$
$s - c = 6{\cdot}75 - 4{\cdot}75 = 2$

$$\therefore \text{true area} = \sqrt{6{\cdot}75 \times 2{\cdot}5 \times 2{\cdot}25 \times 2} \text{ sq. in.}$$
$$= \sqrt{75{\cdot}9375} \text{ sq. in.}$$
$$= 8{\cdot}714 + \text{ sq. in.}$$

$$\text{Computed area} = \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. in.}$$

where $a = 3{\cdot}25$,
$b = 4{\cdot}5$,
$c = 4{\cdot}75$;

$$\therefore s = \frac{3{\cdot}25 + 4{\cdot}5 + 4{\cdot}75}{2}$$
$$= \frac{12{\cdot}5}{2} = 6{\cdot}25$$

and $s - a = 6{\cdot}25 - 3{\cdot}25 = 3$
$s - b = 6{\cdot}25 - 4{\cdot}5 = 1{\cdot}75$
$s - c = 6{\cdot}25 - 4{\cdot}75 = 1{\cdot}5$

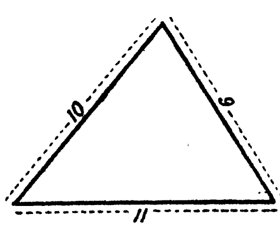
$$\therefore \text{computed area} = \sqrt{6{\cdot}25 \times 3 \times 1{\cdot}75 \times 1{\cdot}5} \text{ sq. in.}$$
$$= \sqrt{49{\cdot}21875} \text{ sq. in.}$$
$$= 7{\cdot}015 + \text{ sq. in.}$$

$$\therefore \text{error} = \text{true area} - \text{computed area}$$
$$= \{(8{\cdot}714+) - (7{\cdot}015+)\} \text{ sq. in.}$$
$$= 1{\cdot}69 + \text{ sq. in.}$$

Example 6.—The sides of a triangle are in the proportion of the numbers 9, 10, 11; and the perimeter is 300 ch.: find its area correct to the nearest square chain.

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. ch.} \quad . \ \S\ 23.$$

where $a = \dfrac{9}{9 + 10 + 11} \times 300$
$= \frac{9}{30} \times 300$
$= 90$
$b = \frac{10}{30} \times 300$
$= 100$
$c = \frac{11}{30} \times 300$
$= 110$
$s = \frac{300}{2}$
$= 150$



$$
\begin{aligned}
\therefore \text{ area of triangle} &= \sqrt{150 \times 60 \times 50 \times 40} \text{ sq. ch.} \\
&= \sqrt{100^2 \times 5^2 \times 3^2 \times 2^2 \times 2} \text{ sq. ch.} \\
&= 100 \times 5 \times 3 \times 2\sqrt{2} \text{ sq. ch.} \\
&= 3000\sqrt{2} \text{ sq. ch.} \\
&= 3000 \times 1{\cdot}414 \text{ sq. ch. nearly} \\
&= 4242 \text{ sq. ch. nearly}
\end{aligned}
$$

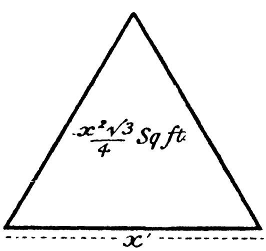
Example 7.—What is the side of that equilateral triangle whose area costs as much to pave at 10 annas a square foot as it would cost to fence the three sides at Rs.12 a yard?

Let each side measure x ft.

Then the area of the triangle will measure $\frac{x^2\sqrt{3}}{4}$ sq. ft. . . § 21.

$\therefore$ cost of paving at 10 annas a sq. ft. $= \frac{10 \times x^2\sqrt{3}}{4}$ annas

Again, perimeter of triangle $= 3x$ ft.



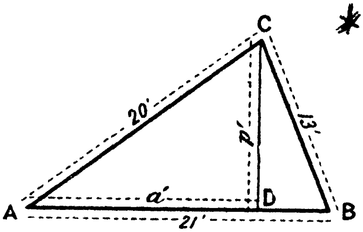
$\therefore$ cost of fencing at Rs.12 a yard or Rs.4 a foot $= 4 \times 3x$ Rs.

$= 16 \times 4 \times 3x$ annas

But cost of paving $=$ cost of fencing

$$
\begin{aligned}
\therefore \frac{10 \times x^2\sqrt{3}}{4} &= 16 \times 4 \times 3x \\
\text{or } x &= \frac{128\sqrt{3}}{5} \\
&= \frac{128 \times 1{\cdot}73205+}{5} \\
&= 44{\cdot}34+
\end{aligned}
$$

Hence side measures $44{\cdot}34+$ ft.



* *Example* 8.—The sides of a triangle are 21 ft., 20 ft., and 13 ft. respectively: find the areas of the triangles into which it is divided by the perpendicular upon the longest side from the opposite angular point.

Let ABC be a triangle having $AB = 21$ ft., $AC = 20$ ft., and $BC = 13$ ft.

Then—

$$
\begin{aligned}
\text{Area of triangle} &= \sqrt{27 \times 6 \times 7 \times 14} \text{ sq. ft.} \quad . \quad . \quad . \ \S\ 23. \\
&= 126 \text{ sq. ft.}
\end{aligned}
$$

Again, if p ft. denote the measure of CD—

$$
\begin{aligned}
\text{area of triangle} &= \tfrac{1}{2} \times p \times 21 \text{ sq. ft.} \quad . \quad . \quad . \quad . \ \S\ 20. \\
\therefore \tfrac{1}{2} \times p \times 21 &= 126 \\
\therefore p &= 12
\end{aligned}
$$

Again, let AD measure a ft.

$$\text{Then } a = \sqrt{(20)^2 - (12)^2} \quad . \quad . \quad . \text{ § 16.}$$
$$= 16$$
$$\text{Hence area of } \triangle ACD = \tfrac{1}{2} \times 16 \times 12 \text{ sq. ft.} \quad . \quad . \text{ § 20.}$$
$$= 96 \text{ sq. ft.}$$
$$\text{and area of } \triangle BCD = \triangle ABC - \triangle ACD$$
$$= (126 - 96) \text{ sq. ft.}$$
$$= 30 \text{ sq. ft.}$$

Examples—V. A.

Find the areas of the following triangles, having—

1. Base 23 ft., height 16 ft. (give the result in square feet).

2. Base 3 yds. 2 ft., height 4 yds. 1 ft. (give the result in square yards and square feet).

3. Base 4 yds. 1 ft. 9 in., height 3 yds. 2 ft. 7 in. (give the result in square yards, square feet, and square inches).

4. Base 13 ch. 75 lks., height 9 ch. 30 lks. (give the result in acres).

Find the heights of the following triangles, having—

5. Area 72 sq. ft., base 16 ft. (give the result in feet).

6. Area 52 sq. yds. 18 sq. in., base 8 yds. 2 ft. 3 in. (give the result in yards, feet, and inches).

7. Area 34 ac., base 8 ch. 30 lks. (give the result in chains and links).

8. Area 22·8 ac., base 936 lks. (give the result in chains).

Find the areas of the following right-angled triangles, having—

9. Hypotenuse 85 in., side 68 in. (give the result in square feet and square inches).

10. Hypotenuse 4 yds. 1 ft. 6 in., side 2 yds. 1 ft. 4 in. (give the result in square yards).

11. Hypotenuse 7 ch. 25 lks., side 6 ch. 44 lks. (give the result in acres).

12. Hypotenuse 4 yds. 1 ft. 1 in., side 2 yds. 1 ft. 1 in. (give the result in square yards, square feet, and square inches).

Find the areas of the following equilateral triangles, having—

13. Side 17 ft. (give the result in square feet).

14. Side 3 yds. 2 ft. 9 in. (give the result in square yards, square feet, and square inches).

15. Side 6 ch. 40 lks. (give the result in square chains).

Find the areas of the following triangles, whose sides are—

16. 25, 17, 12 ft.

17. 132, 125, 37 lks.

18. 195, 364, 533 in.

19. 425, 867, 1258 ft.

20. 1001, 1540, 1617 lks.

21. 4, 7·4, 10·2 ch.

22. Find the equal sides of an isosceles triangle whose base measures 28 in., and whose area measures 672 sq. in.

23. If the sides of a triangle are 35, 44, and 75 ch., find the perpendicular distance of its longest side from the opposite corner.

24. The perimeter of an equilateral triangle is 159 lks. : find its area.

25. Find the rent per acre of a triangular field, which is let for Rs.40 a month, if its sides measure 9·5, 22·8, 24·7 ch. respectively.

26. The area of an equilateral triangle is 24 sq. ft. : find the length of a side correct to the nearest inch.

27. The sides of a triangular piece of ground measure 455, 455, 784 yds. respectively: find the number of trees which can be planted in it, if each tree occupies 4 sq. yds.

28. The sides of a triangle are in the proportion of the numbers 4, 5, 6, and the perimeter is 195 ft.: find its area correct to the hundredth part of a square foot.

29. The perimeter of an equilateral triangle measures as many yards as the area of the equilateral triangle measures square yards: find the length of a side.

30. A plot of ground is in the form of an isosceles triangle. If it cost Rs.1000 at the rate of Rs.2 8 annas per square yard, and if each of the equal sides measure 40 yds., find the length of the base.

31. The diagonal of a square is three times the side of an equilateral triangle: find the ratio of the area of the square to the area of the triangle.

Examples—V. B.

Find the areas of the following triangles, having—

32. Base 15 rasi, height 13 rasi (give the result in bighas).

33. Base 5 rasi 8 lathas, height 4 rasi 14 lathas (give the result in bighas and biswas).

34. Base 4 rasi 12 lathas, height 5 rasi 16 lathas (give the result in bighas, biswas, and biswansi).

35. Find the base of a right-angled triangle whose area measures 2 bighas, and whose perpendicular measures 25 lathas.

36. Find the height of a triangle whose area measures 1 biswa, and whose base measures 5 lathas.

37. Find the base of a triangle whose height measures 2 rasi 4 lathas, and whose area measures 5 bighas 8 biswas (give the result in lathas).

38. Find the area in bighas of an equilateral triangle whose side measures 3 rasi.

39. Find the area in biswas and biswansi of a triangle whose sides measure 14, 48, and 50 lathas respectively.

Examination Questions—V.

A. *Allahabad University: Matriculation.*

1. The sides of a triangle are 25, 39, 56 ft. respectively: find the perpendicular from the opposite angle on the side of 56 ft.

2. (*a*) What is meant by "area"?

(*b*) The area of an acute-angled triangle is 336 sq. ft., and the sides are 26 ft. and 30 ft.: find the base.

3. The three sides AB, AC, BC of the triangle ABC are 68, 75, and 77 ft. respectively: find the length of the perpendicular from A on BC.

4. The area of an equilateral triangle is 25 sq. in.: find its perimeter.

5. Find the least possible length of fencing that can include a triangular area of 10 sq. ft.

6. A man observes the elevation of the top of a tower to be 60°. He then walks a distance of 300 ft., takes a turn of a right angle, and, after walking 400 ft. more, finds he is on the other side of the tower, opposite to his original position. The elevation of the tower is now found to be 30°: find the height of the tower.

B. *Punjab University: Matriculation.*

7. The sides of a triangle are 13, 14, and 15 ft.: find the perpendicular from the opposite angle on the side of 14 ft.

8. The sides of a triangle are 7, 24, and 25 ft. respectively: find the area.

9. Give the rule for finding the area of a triangle in terms of its sides. The sides of a triangle are $2\frac{1}{4}$, 3, $3\frac{3}{4}$ ft.: find in inches the area of the triangle.

C. *Punjab University: Middle School*

10. Two sides of a triangle are 85 and 154 ft. respectively, and the perimeter is 324 ft.: find the area of the triangle.

11. The side of an equilateral triangle is 7 ft.: find the area.

D. *Calcutta University: Matriculation.*

12. The sides of a triangle are 18, 20, and 22 respectively: calculate its area to three places of decimals.

13. The sides BC, CA, AB of the triangle ABC are 13, 12, and 5 respectively, and D is the middle point of BC: find the area of the triangle ABC, and the length of the line AD.

14. The area of a triangular field is 2 ac. 3 ch.; the line drawn from the vertex of the same perpendicular to the base measures 13 po. or per.: what is the length of the base line in chains and links?

E. *European Schools: Final. United Provinces.*

15. A house 42 ft. wide has a roof with unequal slopes, the lengths of which are 26 and 40 ft.: find the height of the ridge above the eaves.

16. The sides of a triangle are 143, 407, and 440 yds. respectively: find the rent of the field at £2 3*s*. per acre.

17. Find the area of a triangle the sides of which are 20, 493, and 507 yds. respectively.

F. *Madras Technical: Elementary.*

18. The side of an equilateral triangle is 10 ft.: find the area in square feet.

19. An equilateral triangle measures 362 sq. ft.: find the length of one side.

G. *Madras Technical: Intermediate.*

20. Determine the area of a triangular plot of ground whose sides are 8 ft., 10 ft., 12 ft.

21. An equilateral triangle measures 1 ac.: find the length of a side in feet.

H. *Roorkee Engineer: Entrance.*

22. Find the area of a triangular field whose sides are 1200, 1800, and 2400 lks. (answer to be given in acres, roods, and perches).

23. An acre and a half of land, in the form of a right-angled triangle, is divided into two parts by a line which bisects the right angle, and which measures $82\frac{1}{2}$ yds.: find the two areas.

24. Find the area in acres, roods, and perches of a field whose sides are 848, 900, and 988 lks.

25. What is the side of that equilateral triangle whose area cost as much paving at 8*d*. a ft. as railing the three sides did at a guinea a yard?

26. Given two sides of an obtuse-angled triangle which are 20 and 40 po., find the third side, that the triangle may contain just one acre of land.

27. The sides of a triangle are 51, 52, 53 ft.: find the perpendicular from the opposite angle on the side of 52 ft., and find the areas of the two triangles into which the original triangle is divided.

28. A rectangular field is 1200 yds. long and 115 yds. broad: find the length of a fence running from one corner to the opposite side that will cut off 3 ac. of ground.

29. The sides of a triangle are in the ratio of 13, 14, 15, and the perimeter is 84 yds. : find the perpendiculars from the angular points upon the sides.

30. The sides of a triangular field are 191, 245, and 310 ft. : find the area in acres.

I. *Roorkee Upper Subordinate : Entrance.*

31. The three sides of a triangle are 800, 500, and 1237 lks. By some mistake the third side was also put down as 500 instead of 1237 : what error would that mistake occasion in the computed area?

32. Find the area in acres, roods, and perches of a triangle whose sides measure 405, 378, and 351 ft.

33. What is the side of an equilateral triangle which has as many square yards in its area as lineal yards in its periphery?

34. The sides of a triangle are 4789, 3742, and 2987 ft. : find the area in yards.

35. From a point within an equilateral triangle perpendiculars are drawn to the three sides, and are 8, 10, and 12 ft. respectively : find the side and the area of the triangle.

36. A garden containing 1 ac. is in the form of a right-angled isosceles triangle. A walk passing round it at 6 ft. from the boundary wall occupies one-fourth of the whole garden : find the width of the walk.

37. The base of a triangular field is 1210 yds., and the height is 496 yds. ; the field is let for £248 a year : find at what price per acre the field is let.

38. The sides of a triangle are in the proportion of the numbers 13, 14, 15, and the perimeter is 50 yds. : find the area.

39. The side of a square is 100 ft. ; a point is taken inside the square which is distant 60 ft. and 80 ft. respectively from the two ends of a side : find the areas of the four triangles formed by joining the point to the four corners of the square.

40. The sides of a triangle are 1137, 1259, and 1344 ft. : find the area in acres, roods, and perches.

41. What is the area of a triangle whose sides are 165, 220, and 275 ft.? Find the answer in acres, roods, and perches.

J. *Roorkee Engineer: Final.*

42. Find the side of an equilateral triangle whose area is 5 ac. (give the answer in feet).

43. The perimeter of an isosceles triangle is 306 ft., and each of the equal sides is $\frac{5}{8}$ of the base : find the area.

44. A triangular field, whose sides measure 375, 300, and 225 yds., is sold for £8500 : find the price per acre.

45. The area of an equilateral triangle is 1943·737 sq. ft. : find its side.

46. The sides of a triangle, of which the perimeter measures 462 ft., are in the ratio of 6, 7, and 8 : find its area.

47. A triangular field is let for £5 11*s.* 6½*d.*, at the rate of £12 an acre. One side is 738 lks. : find the perpendicular on this side from the opposite angle.

48. Find the cost of painting the gable end of a house at 1*s.* 9*d.* per square yard, the breadth being 27 ft., the distance of the eaves from the ground 33 ft., and the perpendicular height of the roof 12 ft.

K. *Roorkee Upper Subordinate: Monthly.*

49. The sides of a triangular field are 350, 440, and 750 yds. ; the field is let for £26 5*s.* a year : find at what price per acre the field is let.

50. The sides of a triangle are 35, 39, and 56 ft. respectively : find the

areas of the two triangles into which it is divided by the perpendicular from the opposite angle on the largest side.

51. The paving of a triangular court came to £100, at 1*s.* 3*d.* per square foot: if one of the sides be 24 yds. long, find the length of the other two equal sides.

52. The sides of a triangle are 15, 14, and 13 ft.: find the area in square links.

53. One side of a triangular court is 98 ft., and the perpendicular on it from the opposite angle is 63 ft.: required the expense of paving it at Rs. 1 3 annas per square yard.

54. The sides of a triangle are 1200, 1450, and 1650 ft.: find the area in square yards.

55. The sides of a triangle are 1115, 1750, and 1765 ft.: find the area in acres, roods, and perches.

56. The sides of a triangle are in the proportion of 13, 14, and 15, and the perimeter is 70 yds.: find the area.

57. What must be the side of an equilateral triangle so that its area may be equal to that of a square of which the diagonal is 120 ft.?

58. A triangular field, 363 yds. long and 240 yds. in the perpendicular, produces an income of £36 a year: at how much an acre is it let?

59. A field, whose three sides are equal, cost Rs. 55 6 annas 9 pies turfing at the rate of 5 annas per 100 square feet: find the length of one of its sides.

L. *Roorkee Upper Subordinate: Final.*

60. In a place where land costs £40 an acre, a triangular field was bought for £300, of which one side measured 302 yds. 1 ft. 6 in.: what was the height of this triangle in yards?

M. *Sandhurst.*

61. Find the area of an isosceles triangle whose base is 16 ft. long, and sides each 17 ft. long.

62. Find (correct to the thousandth part of an inch) the length of one of the equal sides of an isosceles triangle on a base of 14 in., having an area of 92·4 sq. in.

N. *Militia: Literary.*

63. If the length of each side of an equilateral triangle were increased by 1 ft., the area would be increased by $\sqrt{3}$ sq. ft.: find the length of each side.

O. *Additional Examination Questions.—V.* (For Answers, see p. 167.)

64. A person standing at a point A due south of a tower observes the altitude of the tower to be 60°. He then walks to a point B due west of A, and observes the altitude to be 45°; and again at a point C in AB produced he observes the altitude to be 30°: show that B is midway between A and C. (Calcutta University: F.E. Examination.)

65. The sides of a triangle are 25, 101, 114: find the two parts into which the longest side is divided by the perpendicular from the opposite angle. (European Schools: Final. U.P.)

66. The sides of a triangle are 17, 15, and 8 in. respectively: find the length of the straight line joining the middle point of 17 to the opposite angle. (Allahabad University: Matriculation.)

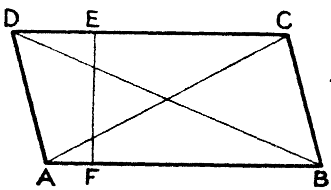
67. Find in acres the area of a triangle whose sides are $\frac{121}{3}\sqrt{6}$, $101\sqrt{24}$, $725\sqrt{\frac{2}{3}}$ yds. respectively. (Punjab University: Matriculation.)

68. The medians of a triangle are 105, 156, 219 ft. respectively: find the area of the triangle. (European Schools: Final. U.P.)

CHAPTER VI.

ON PARALLELOGRAMS.

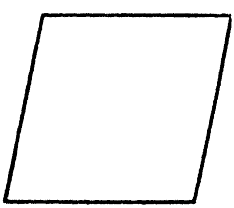
26. A *parallelogram* is a four-sided rectilineal figure having its opposite sides parallel



The *diagonals* (or diameters) of a parallelogram are the straight lines joining opposite angular points.

The side of a parallelogram on which it may be supposed to stand is called its *base.*

The *height* of a parallelogram is the perpendicular distance between its base and the side opposite to the base.

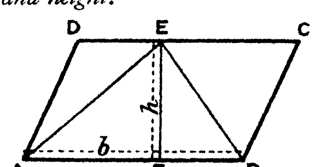


Thus, in the parallelogram $ABCD$, AC and BD are the two diagonals. AB may be regarded as the base, EF as the height.

When the parallelogram is equilateral it is called a *rhombus* (see fig.).

PROPOSITION V.

27. *To find the area of a parallelogram, having given its base and height.*



Let $ABCD$ be a parallelogram.

Let its base AB and its height EF measure b and h of the same linear unit respectively.

It is required to find the area of the parallelogram in terms of b and h.

Join AE and EB.

Now, since—

Area of parallelogram $ABCD = 2 \times$ area of $\triangle AEB$ Euc. I. 41.
$\therefore$ area " " $ABCD = 2 \times \frac{1}{2} . AB \times EF$ . . § 20.
$= bh$ sq. units

Hence rule—

The number of any linear unit in the base of a parallelogram multiplied by the number of the same linear unit in the height gives the number of the corresponding square unit in the area.

Or briefly—

The area of a parallelogram = base × height

$$A = bh \quad . \quad . \quad . \quad \text{(i.)}$$

∴ the base of a parallelogram $= \dfrac{\text{area}}{\text{height}}$

$$b = \frac{A}{h} \quad . \quad . \quad . \quad \text{(ii.)}$$

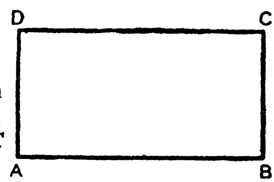
and the height of a parallelogram $= \dfrac{\text{area}}{\text{base}}$

$$h = \frac{A}{b} \quad . \quad . \quad . \quad \text{(iii.)}$$

PARTICULAR CASE.

28. Rectangle.

Let $ABCD$ be a rectangle.



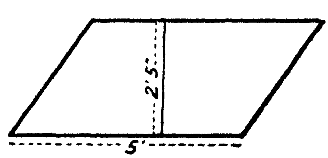
Then, if AB be regarded as its base, BC will evidently be its height.

Now, area of any parallelogram
$=$ base × height . . . . § 27.
$\therefore$ area of rectangle $ABCD = AB \times BC$
$=$ length × breadth
$A = a \times b$

This result has previously been obtained in § 8.

ILLUSTRATIVE EXAMPLES.

29. *Example* 1.—The base of a parallelogram is 5 ft., and its height is 2 ft. 5 in. : find its area in square inches.

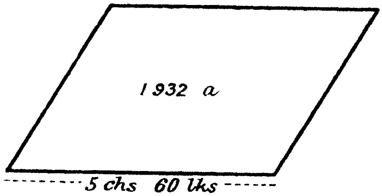


Area of parallelogram $= bh$ sq. in. § 27.

where $b = 5 \times 12 = 60$
and $h = 2 \times 12 + 5 = 29$

$\therefore$ area of parallelogram $= 60 \times 29$ sq. in.
$= 1740$ sq. in.

Example 2.—The area of a parallelogram is 1·932 ac. If its base is 5 ch. 60 lks., what is its height?



$$\left.\begin{array}{r}\text{Height of}\\ \text{parallelogram}\end{array}\right\} = \frac{A}{b} \text{ ch.} \quad . \ \S\ 27$$

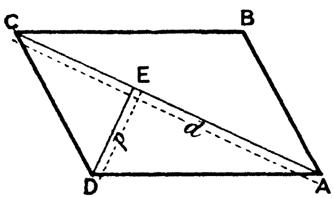
where $A = 1·932 \times 10 = 19·32$
and $b = 5·6$

$$\therefore \left.\begin{array}{r}\text{height of}\\ \text{parallelogram}\end{array}\right\} = \frac{19·32}{5·6} \text{ ch.}$$
$$= 3·45 \text{ ch.}$$
$$= 3 \text{ ch. } 45 \text{ lks.}$$

PROPOSITION VI.

30. *To find the area of a parallelogram, having given a diagonal. and the perpendicular distance of this diagonal from either of the outlying vertices.*

Note.—In any parallelogram, it may be proved by geometry that the perpendiculars drawn from one pair of opposite vertices to the diagonal joining the other pair are equal.



Let $ABCD$ be a parallelogram. Let its diagonal AC measure d of any linear unit. Let DE, the perpendicular from D on AC, measure p of the same linear unit.

It is required to find the area of the parallelogram in terms of d and p.

Now, since—

Area of parallelogram $ABCD = 2 \times$ area of $\triangle ADC$ Euc. I. 34.
$\therefore$ area „ „ $ABCD = 2 \times \frac{1}{2} AC \times DE$ . . § 20.
$= dp$ sq. units

Hence rule—

The number of any linear unit in the diagonal of a parallelogram multiplied by the number of the same linear unit in the perpendicular distance of this diagonal from either of the outlying angular points gives the number of the corresponding square unit in the area.

Or briefly—

Area of parallelogram = diagonal × perpendicular distance of this diagonal from either of the outlying vertices

$$\mathbf{A = dp} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (\text{i.})$$

$$\therefore \textbf{diagonal} = \frac{\textbf{area}}{\textbf{perpendicular distance}}$$

$$\mathbf{d = \frac{A}{p}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (\text{ii.})$$

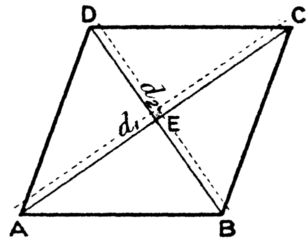
and perpen. distance $= \dfrac{\textbf{area}}{\textbf{diagonal}}$

$$\mathbf{p} = \frac{\mathbf{A}}{\mathbf{d}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (iii)$$

PARTICULAR CASE.

31. Rhombus.

Let $ABCD$ be a rhombus.



It can be proved by geometry that its diagonals AC and BD bisect one another at right angles.

That is—

$$DE = \tfrac{1}{2} \cdot DB = p$$

and if d_1, d_2 express the measures of AC and DB in terms of the same linear unit—

$$p = \tfrac{1}{2}DB = \frac{d_2}{2}$$

$$\text{And area of rhombus} = AC \times DE \quad . \; . \qquad \S\ 30.$$

$$= d_1 \times \frac{d_2}{2} \text{ square units}$$

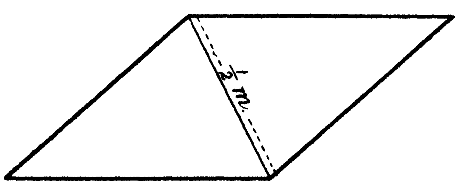
Hence rule—

Area of rhombus $= \frac{1}{2} \times$ **product of its diagonals**

$$\mathbf{A} = \tfrac{1}{2}\mathbf{d}_1\mathbf{d}_2$$

ILLUSTRATIVE EXAMPLES.

32. *Example* 1.—The area of a tract of country in the form of a parallelogram is 200 ac., and the length of one of its diagonals is $\frac{1}{2}$ mile : find the distance of this diagonal from either of the outlying corners.



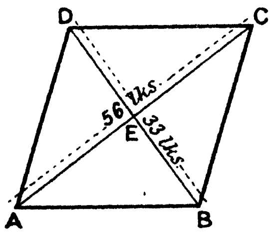
$$\text{Required distance} = \frac{A}{d} \text{ yds.} \quad . \; . \; . \; . \; . \; . \quad \S\ 30.$$

where $A = 200 \times 4840$,
 and $d = 880$;

$$\therefore \text{ required distance} = \frac{200 \times 4840}{880} \text{ yds.}$$

$$= 1100 \text{ yds.}$$

Example 2.—The diagonals of a rhombus are 56 lks. and 33 lks. respectively : find the length of a side.



In the rhombus $ABCD$ let AC measure 56 lks., and BD 33 lks.

Now, because the diagonals of a rhombus bisect one another at right angles, ... § 31.

∴ in the $\triangle AEB$—

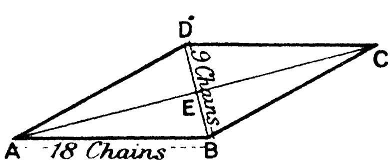
$\angle AEB$ is a right angle
$AE = \frac{1}{2}.AC = 28$ lks.
$BE = \frac{1}{2}.BD = 16{\cdot}5$ lks.

Hence—

$$AB = \sqrt{(28)^2 + (16{\cdot}5)^2} \text{ lks.} \quad . \ . \ . \ . \ . \ \S\ 16.$$
$$= 32{\cdot}5 \text{ lks.}$$

Example 3.—The side of a rhombus $ABCD$ is 18 ch., and one of its diagonals BD is 9 ch. : find the other diagonal, and the area of the rhombus.

∵ AEB is a right-angled triangle, ... § 31.



$$\therefore AE = \sqrt{AB^2 - BE^2} \quad \S\ 16.$$
$$= \sqrt{(18)^2 - (\tfrac{9}{2})^2} \text{ ch.}$$
$$= \sqrt{\frac{1296 - 81}{4}} \text{ ch}$$
$$= \frac{\sqrt{1215}}{2} \text{ ch.}$$
$$= \frac{34{\cdot}856\ldots}{2} \text{ ch.}$$

$$\therefore AC = 34{\cdot}856\ldots \text{ ch.}$$

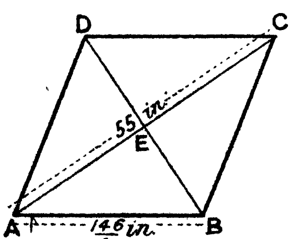
Again—

Area of rhombus $= \frac{1}{2}d_1d_2$ sq. ch. . . . . . § 31.

where $d_1 = 9$,
and $d_2 = 34{\cdot}856$ . . .

∴ area = $156{\cdot}85$ . . . sq. ch.

* *Example* 4.—The perimeter of a rhombus is 146 in., and one of its diagonals is 55 in. : find the other diagonal.



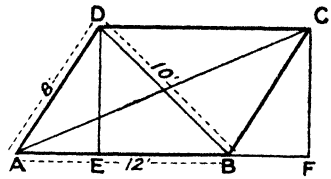
Let $ABCD$ be a rhombus in which AC measures 55 in., and let its perimeter measure 146 in.

Then AB measures $36{\cdot}5$ in.
and AE " $27{\cdot}5$ in.

$$\therefore BE = \tfrac{1}{2}.BD = \sqrt{AB^2 - AE^2} \quad \S\ 16.$$
$$= \sqrt{(36{\cdot}5)^2 - (27{\cdot}5)^2} \text{ in.}$$
$$= 24 \text{ in.}$$

Hence other diagonal = 48 in.

Example 5.—A parallelogram, the lengths of whose sides are 12 ft. and 8 ft., has one diagonal 10 ft. long: find the length of the other diagonal.



In the parallelogram $ABCD$, let $AB = 12$ ft., $AD = 8$ ft., $BD = 10$ ft.

From D and C draw DE and CF perpendiculars on AB and AB produced respectively.

Now—

$$DB^2 = DA^2 + AB^2 - 2 \,.\, AB \,.\, AE \quad . \quad . \text{ Euc. II. 13.}$$

Also—

$$AC^2 = BC^2 + AB^2 + 2 \,.\, AB \,.\, BF \quad . \quad . \text{ Euc. II. 12}$$

Therefore, adding, we get—

$$DB^2 + AC^2 = 2 \,.\, AD^2 + 2AB^2 \text{ (since } AE = BF\text{)}$$

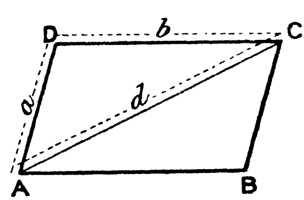
Hence, if AC measure x ft —

$$\begin{aligned} x^2 &= 2 \times (8)^2 + 2 \times (12)^2 - (10)^2 \\ &= 128 + 288 - 100 \\ &= 316 \\ x &= \sqrt{316} \\ &= 17\cdot776 \ldots \end{aligned}$$

Hence other diagonal measures 17·776 ft. nearly.

PROPOSITION VII.

33. *To find the area of a parallelogram, having given a diagonal and two adjacent sides.*



Let $ABCD$ be a parallelogram.

Let its diagonal AC measure d of any linear unit. Let its two adjacent sides AD, DC measure a and b of the same linear unit respectively.

Then a and b will also denote the measures of BC and AB. Euc. I. 34.

It is required to find the area of the parallelogram in terms of a, b, and d.

Now, since—

Area of parallelogram $ABCD = 2 \times$ area of $\triangle ADC$. Euc. I. 34.

$\therefore$ area of parallelogram $ABCD = 2 \times \sqrt{s(s-a)(s-b)(s-d)}$ sq. units . . . . § 23.

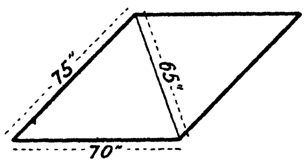
Hence rule—

The area of a parallelogram is equal to twice the area of a triangle having for its sides a diagonal of the parallelogram and any two of its adjacent sides.

ILLUSTRATIVE EXAMPLE.

✱ **34.** *Example.*—Find the area of a parallelogram of which a diagonal measures 65 in., and of which two adjacent sides measure 70 in. and 75 in. respectively.

Area of parallelogram $= 2\sqrt{s(s-a)(s-b)(s-d)}$ sq. in. . § 33.



where $a = 70$,
$b = 75$,
$d = 65$;

$$\therefore s = \frac{70 + 75 + 65}{2} = 105$$

and $s - a = 105 - 70 = 35$
$s - b = 105 - 75 = 30$
$s - d = 105 - 65 = 40$

$$\begin{aligned}\therefore \text{ area of parallelogram } &= 2 \times \sqrt{105 \times 35 \times 30 \times 40} \text{ sq. in.}\\ &= 2 \times \sqrt{5^2 \times 7^2 \times 3^2 \times 10^2 \times 2^2} \text{ sq. in.}\\ &= 2 \times 2100 \text{ sq. in.}\\ &= 4200 \text{ sq. in.}\end{aligned}$$

Examples—VI. A.

Find the areas of the following parallelograms :—

1. Base 24 ft., height 13 ft.
2. Base 5 yds. 2 ft., height 9 yds. 1 ft.
3. Base 15·46 ch., height 12·72 ch.

Find the bases of the following parallelograms :—

4. Area 256 sq. ft., height 32 ft.
5. Area 23 sq. yds. 8 sq. ft., height 14 yds. 1 ft.
6. Area 1 ac., height 3·8 ch.

Find the areas of the following parallelograms :—

7. One diagonal 5 ft. 8 in. ; perpendicular distance of this diagonal from either of the outlying vertices, 2 ft. 3 in.

8. One diagonal 15 ch. 36 lks. ; perpendicular distance of this diagonal from either of the outlying vertices, 7 ch. 54 lks. (express the result in acres).

9. The area of a parallelogram measures 15 sq. yds , and one of its diagonals measures 7 yds. 1 ft. 6 in. : find the distance of this diagonal from the outlying vertices.

10. Find the perimeter of a rhombus whose diagonals measure 3 ft. 6 in. and 2 ft. 9 in. respectively.

11. Find the cost, at 4 annas 6 pies a square yard, of turfing a plot of ground in the form of a parallelogram whose base measures 62 ft., and height 46 ft.

12. Find the area of a rhombus whose perimeter is 248 in., and one of whose diagonals is 104 in.

13. The side of a rhombus is 65 ch., and one of its diagonals is 112 ch. : find the length of the other diagonal.

Examples—VI. B.

Find the areas of the following parallelograms :—

14. Base 20 rasi, height 16 rasi.

15. Base 10 rasi 6 lathas, height 32 lathas.

16. Base 8·3 rasi, height 6·8 rasi.

17. Find the base of a parallelogram whose area measures 2 bighas, and whose height measures 3 rasi 6 lathas.

18. Find the area of a parallelogram, one of whose diagonals measures 2 rasi 8 lathas, and the perpendicular distance of this diagonal from one of the outlying vertices measures 10 lathas.

19. Find the area of a rhombus whose diagonals measure 15 lathas and 13 lathas respectively.

Examination Questions—VI.

A. *Allahabad University: Matriculation.*

1. The diagonals of a rhombus are 6 ft. and 8 ft. : find the side and the height.

B. *Calcutta University: Matriculation.*

2. The diagonals of a rhombus are 72 and 96 : find its area and the lengths of its sides.

C. *Madras Technical: Elementary.*

3. Each side of a rhombus is 330 ft., and one diagonal is 500 ft. : find the area of the rhombus in acres and cents.

4. Find the area of a rhombus in square feet, the diagonals being 160 ft. and 100 ft.

5. The area of a mat in the form of a rhombus is 8 sq. yds., and the perimeter is 36 ft. : find its perpendicular breadth.

D. *European Schools: Final. United Provinces.*

6. The semi-diagonals of a rhombus are 8 and 16 in. respectively : find the area of the rhombus, and also the length of its side.

E. *Roorkee Engineer: Entrance.*

7. The area of a rhombus is 120,000 ft., and the side 400 : find the diagonals.

F. *Roorkee Upper Subordinate: Entrance.*

8. The diagonals of a rhombus are respectively 40 and 60 yds. : find its area, perimeter, and height.

9. The diagonals of a rhombus are 88 and 234 ft. respectively : find the area ; find also the length of a side and the height of the rhombus.

G. *Roorkee Engineer: Final.*

10. The side of a rhombus is 36 ft., and one of its diagonals is 18 ft. : find the other diagonal and the area of the figure.

11. The side of a rhombus is 20, and its longer diagonal is 34·64 : find the area and the other diagonal.

H. *Roorkee Upper Subordinate: Monthly.*

12. The area of a rhombus is 354,144 sq. ft., and one diagonal is 672 ft.: find the other diagonal; find also the length of a side, and the height of the rhombus.

13. The diagonals of a rhombus are 60 ft. and 45 ft. respectively: find its area; find also the length of a side and the height of the rhombus.

14. The side of a rhombus is 20 ft., and its shorter diagonal is three-fourths the longer one: find its area.

I. *Additional Examination Questions.—VI.* (For Answers, see p. 168.)

15. The diagonals of a rhombus are 4 ft. and 1 ft. 2 in.: find the sides and the area. (Allahabad University: Matriculation.)

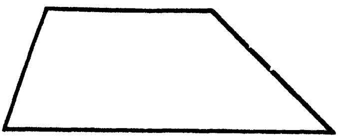
16. A field is in the form of a rhombus whose diagonals are 2870 links and 1850 links: find to the nearest penny the rent at £4 10*s.* 6*d.* an acre. (Calcutta University: F.E. Examination.)

17. The diagonals of a rhombus are 80 and 60 ft. respectively: find the area, length of side, and height of the rhombus. (Roorkee Upper Subordinate: Entrance.)

CHAPTER VII.

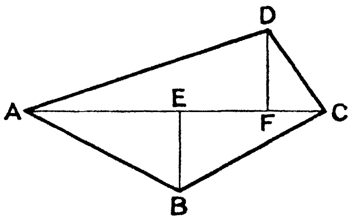
ON ANY QUADRILATERALS.

35. A *quadrilateral* is any figure bounded by four straight lines.



When two of the sides of a quadrilateral are parallel to one another, the figure is called a *trapezoid* (see fig.).

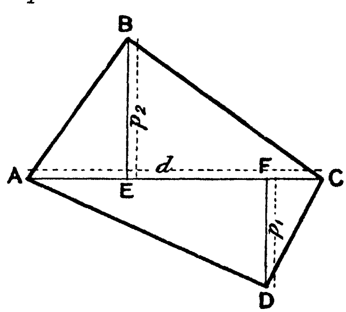
The perpendiculars drawn from two opposite angular points of a quadrilateral to the diagonal joining the remaining angular points are called the *offsets* from that diagonal to those angular points.



Thus in the quadrilateral $ABCD$, DF and BE are the offsets from the diagonal AC to the angular points D and B.

PROPOSITION VIII.

36. *To find the area of a quadrilateral, having given a diagonal and its offsets to the outlying angular points.*



Let $ABCD$ be a quadrilateral.

Let its diagonal AC measure d of any linear unit. Let DF and BE, the offsets from this diagonal to D and B, measure p_1 and p_2 of the same linear unit respectively.

It is required to find the area of the quadrilateral in terms of d, p_1, and p_2.

Now, since—

Area of quadrilateral $ABCD$ = area of $\triangle ADC$ + area of $\triangle ABC$

$\therefore$ " " $ABCD = \frac{1}{2} . DF . AC + \frac{1}{2} . BE . AC$ § 20.

$= (\frac{1}{2} p_1 . d + \frac{1}{2} . p_2 . d)$ sq. units

$= \frac{1}{2} d(p_1 + p_2)$ sq. units

Hence rule—

Multiply the number of any linear unit in the diagonal of a quadrilateral by the sum of the numbers of the same linear unit in the offsets from this diagonal to the outlying angular points; then half the product will give the number of the corresponding square unit in the area.

Or briefly—

Area of a quadrilateral = $\frac{1}{2}$ diagonal × (sum of its offsets)

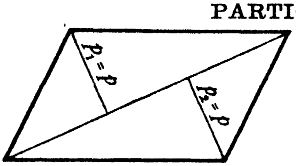
$$A = \tfrac{1}{2} . d(p_1 + p_2) \quad . \quad \text{(i.)}$$

$$\therefore \text{ a diagonal of a quadrilateral} = \frac{2 \times \text{area}}{\text{sum of its offsets}}$$

$$d = \frac{2 \times A}{p_1 + p_2} \quad . \quad . \quad . \quad \text{(ii.)}$$

Note.—If the diagonal falls *outside* the figure, the rule evidently becomes
Area of quadrilateral = $\frac{1}{2}$ diagonal × (difference of its offsets)

PARTICULAR CASES.



37. 1. Parallelogram.

Here the offsets from a diagonal to the outlying angular points are equal to one another . . § 30.

That is, $p_1 = p_2 = p$

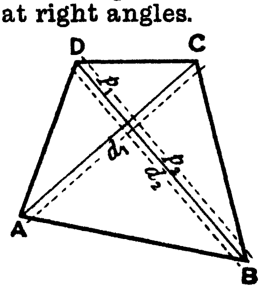
Now, area of any quadrilateral = $\frac{1}{2}d(p_1 + p_2)$ sq. units § 36

$\therefore$ area of parallelogram = $\frac{1}{2} . d(p + p)$ sq. units

= dp sq. units

This result has been previously obtained in § 30.

2. A quadrilateral whose diagonals cut one another at right angles.



Let *ABCD* be a quadrilateral whose diagonals *AC* and *BD* cut one another at right angles.

Let d_1, d_2 be the measures of the diagonals *AC* and *BD*, both expressed in terms of the same unit. The sum of the offsets from one diagonal, *AC*, evidently makes up the other diagonal, *BD*.

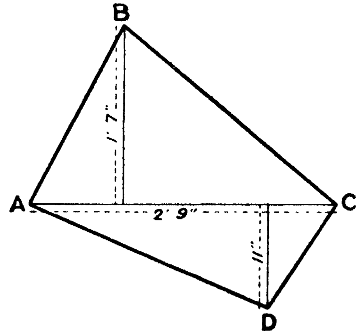
That is, $p_1 + p_2 = d_2$

Now, area of any quadrilateral = $\frac{1}{2}d(p_1 + p_2)$ sq. units . § 36.

$\therefore$ area of quadrilateral *ABCD* = $\frac{1}{2} . d_1 d_2$ sq. units

ILLUSTRATIVE EXAMPLES.

38. *Example* 1.—In a quadrilateral $ABCD$, the diagonal AC measures 2 ft. 9 in., and the offsets from this diagonal to B and D measure 1 ft. 7 in. and 11 in. respectively: find the area of the quadrilateral.

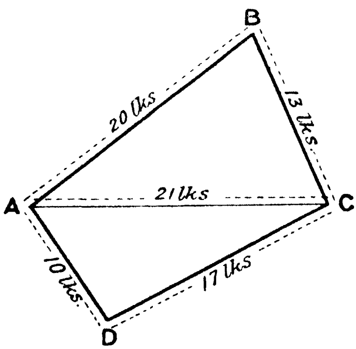


$$\left.\begin{array}{r}\text{Area of}\\ \text{quadrilateral}\end{array}\right\} = \tfrac{1}{2} \cdot d(p_1 + p_2) \text{ sq. in.} \quad . \quad . \quad \S\ 36.$$

where $d = 2 \times 12 + 9 = 33$,
and $p_1 = 1 \times 12 + 7 = 19$,
and $p_2 = 11$;

$$\left.\begin{array}{r}\therefore \text{ area of}\\ \text{quadrilateral}\end{array}\right\} = \tfrac{1}{2} \cdot 33(19 + 11) \text{ sq. in.}$$
$$= \tfrac{1}{2} \cdot 33 \cdot 30 \text{ sq. in.}$$
$$= 495 \text{ sq. in.}$$
$$= 3 \text{ sq. ft. } 63 \text{ sq. in.}$$

Example 2.—In a quadrilateral $ABCD$, the sides AB, BC, CD, DA measure 20, 13, 17, 10 lks. respectively, and the diagonal AC 21 lks.: find the area of the quadrilateral.



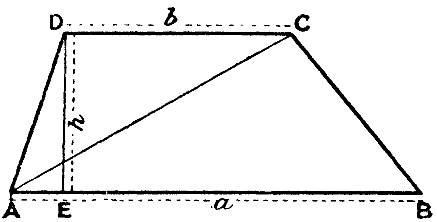
Area of quadrilateral $=$ area of $\triangle\, ABC +$ area of $\triangle\, ACD$
$= \sqrt{27 \times 14 \times 7 \times 6}$ sq. lks. $+ \sqrt{24 \times 14 \times 7 \times 3}$ sq. lks. . . . . § 23.
$=$ 126 sq. lks. $+$ 84 sq. lks.
$=$ 210 sq. lks.

PROPOSITION IX.

39. *To find the area of a trapezoid, having given the parallel sides and the perpendicular distance between them.*

Let $ABCD$ be a trapezoid.

Let its parallel sides AB and DC measure a and b of the same linear unit respectively. Let DE, the perpendicular distance between them, measure h of the same linear unit.



It is required to find the area of the trapezoid in terms of a, b, and h.

Join AC.

Now, since—

$$\text{Area of trapezoid} = \text{area of } \triangle ACD + \text{area of } \triangle ABC$$
$$\therefore \quad ,, \quad ,, \quad = \tfrac{1}{2} . DC \times DE + \tfrac{1}{2} . AB \times DE \qquad \S\ 20.$$
$$= (\tfrac{1}{2} . bh + \tfrac{1}{2}ah) \text{ sq. units}$$
$$= \tfrac{1}{2} . (a + b)h \text{ sq. units}$$

Hence rule—

Multiply the sum of the numbers of any linear unit in the parallel sides of a trapezoid by the number of the same linear unit in the perpendicular distance between them; then half the product will give the number of the corresponding square unit in the area.

Or briefly—

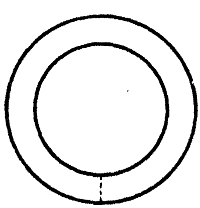
The area of a trapezoid = $\frac{1}{2}$. sum of parallel sides × distance between them

$$\mathbf{A = \tfrac{1}{2}(a + b)h} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\therefore \left.\begin{array}{l}\textbf{the distance between the}\\ \textbf{parallel sides of a trapezoid}\end{array}\right\} = \frac{\mathbf{2 \times area}}{\mathbf{sum\ of\ parallel\ sides}}$$

$$\mathbf{h = \frac{2A}{a + b}} \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

If a ring be cut across at any point in its outer circumference

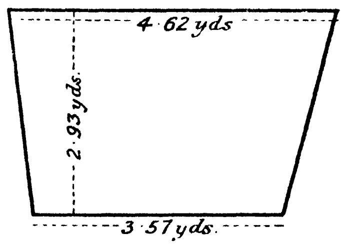


and then straightened out, its surface may be seen to be that of a trapezoid, whose parallel sides are the inner and outer circum-

ferences of the ring respectively, and whose distance between the parallel sides is equal to the width of the ring.

ILLUSTRATIVE EXAMPLES.

40. *Example* 1.—The parallel sides of a trapezoid are 3·57 yds. and 4·62 yds. respectively, and the distance between them is 2·93 yds.: find the area of the trapezoid.

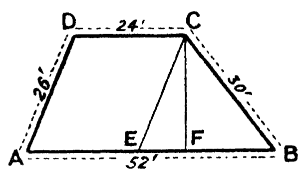


$$\left.\begin{array}{l}\text{Area of}\\ \text{trapezoid}\end{array}\right\} = \tfrac{1}{2}(a+b)h \text{ sq. yds. § 39.}$$

where $a = 3·57$,
$b = 4·62$,
$h = 2·93$;

$$\therefore \left.\begin{array}{l}\text{area of}\\ \text{trapezoid}\end{array}\right\} = \tfrac{1}{2}(3·57 + 4·62) \times 2·93 \text{ sq. yds.}$$
$$= 11·99835 \text{ sq. yds.}$$

Example 2.—The parallel sides of a trapezoid are 24 and 52 ft., and the other sides are 26 and 30 ft.: find the area.



Let $ABCD$ be the trapezoid, so that $DC = 24$ ft., $AB = 52$ ft., $DA = 26$ ft., $CB = 30$ ft.

Through C draw CE parallel to DA, and CF perpendicular to AB.

Then $EB = AB - AE = (52 - 24)$ ft. $= 28$ ft.

Now, area of $\triangle EBC = \sqrt{s(s-a)(s-b)(s-c)}$ sq. ft. § 23.

where $a = 26$,
$b = 28$,
$c = 30$;

$$\therefore s = 42$$
$$\therefore \text{area of } \triangle EBC = \sqrt{42 \times 16 \times 14 \times 12} \text{ sq. ft.}$$
$$= 336 \text{ sq. ft.}$$
$$\text{Also } CF = \frac{2A}{b} \text{ ft. . . . . § 20.}$$

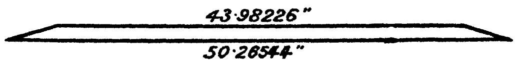
where $A = 336$,
and $b = 28$;

$$\therefore CF = \tfrac{672}{28} \text{ ft.}$$
$$= 24 \text{ ft.}$$
$$\therefore \text{area of trapezoid} = \tfrac{1}{2}(a+b)h \text{ sq. ft.}$$

where $a = 24$,
$b = 52$,
$h = 24$;

$$\therefore \text{area of trapezoid} = \tfrac{1}{2}(24 + 52) \times 24 \text{ sq. ft.}$$
$$= 912 \text{ sq. ft.}$$

Example 3.—Find the area of a plane circular ring whose outer and inner circumferences measure 50·26544 in. and 43·98226 in. respectively, the width of the ring being one inch.



$$\text{Area of ring} = \text{area of trapezoid} \quad . \quad . \quad . \quad . \quad \S\ 39.$$
$$= \tfrac{1}{2}(a+b)h \text{ sq. in.} \quad . \quad . \quad . \quad . \quad \S\ 39.$$

where $a = 50·26544$,
$b = 43·98226$,
$h = 1$;

$$\therefore \text{ area of ring} = \tfrac{1}{2}(50·26544 + 43·98226) \text{ sq. in.}$$
$$= \tfrac{1}{2} \times 94·24770 \text{ sq. in.}$$
$$= 47·12385 \text{ sq. in.}$$

41. The area of a quadrilateral inscribed in a circle may be expressed in terms of the four sides, thus :

$$\text{Area of quadrilateral} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ sq. units}$$

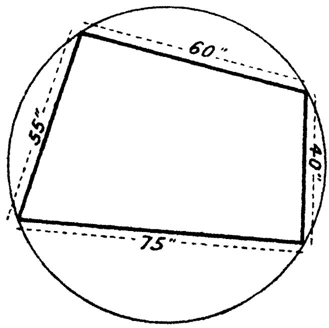
where a, b, c, d are the measures of the sides, all expressed in terms of the same unit, and—

$$s = \frac{a+b+c+d}{2}$$

The proof of this formula depends upon the fact that the opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles. . . . . . . Euc. III. 22.

ILLUSTRATIVE EXAMPLE.

42. *Example.*—The sides of a quadrilateral inscribed in a circle are 75, 55, 60, and 40 in. : find its area.



$$\left.\begin{array}{r}\text{Area of}\\\text{quadl.}\end{array}\right\} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ sq. in.} \quad . \quad . \quad . \quad \S\ 41.$$

where $a = 75$,
$b = 55$,
$c = 60$,
$d = 40$;

$$\therefore s = \frac{75+55+60+40}{2}$$
$$= 115$$

$$\left.\begin{array}{r}\therefore \text{ area of}\\\text{quadrilateral}\end{array}\right\} = \sqrt{40 \times 60 \times 55 \times 75} \text{ sq. in.}$$
$$= 300\sqrt{110} \text{ sq. in.}$$
$$= 3146 \text{ sq. in. nearly}$$
$$= 21·85 \text{ sq. ft. nearly}$$

Examples—VII. A.

Find the areas of the quadrilaterals whose dimensions are as follows :—

1. Diagonal 213 ft., offsets 97 and 103 ft.
2. Diagonal 5 yds. 2 ft., offsets 1 yd. 2 ft. and 2 yds. 1 ft.
3. Diagonal 9 ch. 12 lks., offsets 4 ch. 23 lks. and 5 ch. 56 lks.
4. Diagonal 12 ch. 22 lks., offsets 7 ch. 13 lks. and 8 ch. 24 lks.

Find the areas of the trapezoids whose dimensions are as follows :—

5. Parallel sides 57 and 83 ft., perpendicular distance 16 ft.
6. Parallel sides 4 yds. 1 ft. and 3 yds. 2 ft., perpendicular distance 2 ft.
7. Parallel sides 4 ch. 18 lks. and 7 ch., perpendicular distance 78 lks.
8. Parallel sides 9 ch. 14·3 lks. and 74·5 lks., perpendicular distance 2 ch.

Find the perpendicular distance between the parallel sides of the trapezoids having the following measurements :—

9. Area 340 sq. yds., parallel sides 30 ft. and 18 ft.
10. Area 62 ac. 3 ro., parallel sides 30 ch. and 20 ch.
11. A field is in the form of a trapezoid ; its parallel sides are 9 ch. 50 lks. and 8 ch. 50 lks. ; the perpendicular distance between them is 11 ch. 25 lks. : find the rent at Rs.23 per acre.
12. Find the cost of paving a courtyard in the form of a trapezoid whose parallel sides measure 20 yds. 2 ft. and 17 yds. 1 ft respectively, and the perpendicular distance between them 10 yds., at 7 annas per square foot.
13. Find the cost of a four-sided piece of ground, if one of its diagonals measure 7 ch. 36 lks., and if the offsets from this diagonal to the opposite angular points measure 4 ch. 23 lks. and 6 ch. 19 lks. respectively, at Rs.300 per acre.
14. Find the rent of a four-sided field whose diagonal is 16 ch. 35 lks., and whose offsets are 5 ch. 20 lks. and 7 ch., at Rs.40 per acre.
15. The rent of a four-sided field is Rs.69·3. A diagonal of this field measures 6 ch. 60 lks., and its offsets to the opposite angles measure 5 ch. 10 lks. and 3 ch. 30 lks. respectively : find the rent per acre.
16. A room is in the form of a trapezoid, whose parallel sides measure 35 ft. 7 in. and 24 ft. 5 in. respectively. The perpendicular distance between these sides is 18 ft. What length of matting $\frac{3}{4}$ yd. wide will be required to cover the floor ?
17. Find the area of a quadrilateral whose diagonals are 5 yds. 1 ft. 9 in. and 6 yds. 2 ft. 6 in. respectively, and are at right angles to one another.
18. The difference between the two parallel sides of a trapezoid is 8 ft., the perpendicular distance between them is 24 ft., and the area of the trapezoid is 312 sq. ft. : find the two parallel sides.
19. Find the area of a quadrilateral inscribed in a circle whose sides measure 36, 77, 75, and 40 lks. respectively.

Examples—VII. B.

Find the areas of the quadrilaterals whose dimensions are as follows :—

20. Diagonal 18 rasi, offsets 15 rasi and 21 rasi.
21. Diagonal 3 rasi 8 lathas, offsets 2 rasi and 5 rasi 6 lathas.

Find the areas of the trapezoids whose dimensions are as follows :—

22. Parallel sides 23 rasi and 37 rasi, perpendicular distance 36 rasi.
23. Parallel sides 6 rasi 14 lathas, 4 rasi 12 lathas ; perpendicular distance 3 rasi 4 lathas.

Find the perpendicular distances between the parallel sides of the trapezoids having the following measurements :—

24. Area 3 bighas 10 biswas, parallel sides 1 rasi 15 lathas and 1 rasi 5 lathas.

25. Area 5 bighas 15 biswas, parallel sides 2 rasi 10 lathas and 1 rasi 15 lathas.

Examination Questions—VII.

A. *Allahabad University: Matriculation.*

1. The sides of a quadrilateral inscribed in a circle, taken in order, are 25, 39, 60, and 52 ft.: find the area of the quadrilateral.

2. The opposite sides of a quadrilateral are parallel, and the distance between them is 7 ch. 50 lks.: if the area is 6·75 ac., and the length of one of the parallel sides is 10 ch. 30 lks., find the length of the other.

B. *Punjab University: Matriculation.*

3. Prove the rule for finding the area of a trapezoid.

C. *Punjab University: Middle School.*

4. $ABCD$ is a quadrilateral. Each of the angles ABC and DAC is a right angle; the following lengths are in feet: $AB = 112$, $CD = 175$, $DA = 105$. Find the area.

5. The sides of a quadrilateral, taken in order, are 5, 12, 14, and 15 ft. respectively, and the angle contained by the first two is a right angle: find the area.

6. In a quadrilateral $ABCD$, $AC = 13$ ft., $BD = 12$ ft., AC cuts BD at right angles: find the area.

D. *European Schools: Final. United Provinces.*

7. The sides of a quadrilateral are 75, 75, 100, 100 ft. respectively, and it can be inscribed in a circle: find its area.

8. In a trapezoid the parallel sides are 14 and 20 yds. respectively, and the perpendicular distance between them is 12 yds.: find the area of the trapezoid.

E. *Madras Technical: Elementary.*

9. Find the area of a trapezoid whose parallel sides are 1000 ft. and 1500 ft., and the distance between them 100 ft.

F. *Roorkee Engineer: Entrance.*

10. A field is in shape a trapezoid, whose parallel sides are 6 ch. 75 lks., and 9 ch. 25 lks.: if the area be 2 ac. 3 ro. 8 per., find the shortest way across the field in yards.

11. A field in the form of a quadrilateral, $ABCD$, whose sides taken in order are respectively equal to 192, 576, 288, and 480 ft., has the diagonal AC equal to 672 ft.: find the area in acres, roods, poles, etc.

12. One diagonal of a quadrilateral which falls without the figure is equal to 30 yds., and the difference of the perpendiculars upon it from the remaining angles of the quadrilateral is 14 yds.: find its area.

G. *Roorkee Upper Subordinate: Entrance.*

13. Find in acres the area of a quadrilateral whose diameter is 19·3 ch., and the perpendiculars on which, from the opposite angles, are 13·5 ch. and 18·75 ch. respectively. 1 chain = 66 ft.

14. Find the area in acres of a field $ABCD$. $AD = 220$ yds., $BC = 265$ yds., $AC = 378$ yds., and the perpendiculars from D and B meet the diagonal in E and F, so that $AE = 100$ and $CF = 70$ yds.

15. AC is the diameter of a circle and a diagonal of the inscribed quadrilateral $ABCD$; given $AB = 30$, $BC = 40$, $CD = 10$, find AD and the area of the quadrilateral.

16. How many square yards are there in a trapezoid, the parallel sides of which are 157·6 metres and 94 metres, and the perpendicular distance between them 72 metres? 1 metre = 39·37 in.

17. The area of a trapezoid is 475 sq. ft., the perpendicular distance between the two parallel sides is 19 ft.: find the two parallel sides, their difference being 4 ft.

18. Calculate the area of a trapezoid, the sides of which, taken in order, are 13, 11, 15, and 25, and the second parallel to the fourth.

H. *Roorkee Engineer: Monthly.*

19. How many square yards of paving are there in a quadrangular court whose diagonal is 54 ft., and the perpendiculars on it, from the opposite corners, 25 and $17\frac{3}{4}$ ft. respectively.

20. A trapezoid, with parallel sides of lengths as 3 : 4 is cut from a rectangle $12' \times 2'$, so as to have an area of one-third of the latter: find the lengths of the parallel sides.

I. *Roorkee Engineer: Final.*

21. The parallel sides of a trapezoid are 55 and 77 ft., and the other sides are 25 and 31 ft.: find the area.

22. Two of the four hedges of a field are parallel, and 1000 yds. and 936 yds. respectively. A man standing midway between these parallel hedges observed that a horse he was lunging, with 25 yds. of rope, in crossing the shortest line from his station to either parallel hedge, bisected it: required area of field in acres.

23. A quadrilateral $ABCD$. Find area in acres and decimals of an acre. $AB = 300$ yds., $BC = 350$ yds., $CD = 700$ yds., $DA = 650$ yds., $AC = 400$ yds.

J. *Roorkee Upper Subordinate: Monthly.*

24. The sides of a quadrilateral, taken in order, are 8, 8, 7, 5 ft. respectively, and the angle contained by the first two sides is 60°: find the area.

25. The parallel sides of a trapezoid are 14 and 30 ft. respectively, and the other two sides are 12 and 19 ft.: find the area.

26. One diagonal of a quadrilateral which lies outside the figure is 70 ft., and the difference of the perpendiculars upon it is 16 ft.: find the area.

27. A railway platform has two of its opposite sides parallel, and its other two sides equal; the parallel sides are 100 and 120 ft. respectively, and the equal sides are 15 ft. each: find its area.

28. In a trapezium $ABCD$, $AB = 345$, $BC = 156$, $CD = 323$, $DA = 192$, the diagonal $AC = 438$: find the area.

29. Required the depth of a ditch, the transverse section of which is a trapezoid, area 146·25, breadth at top = 20, side slopes 3 to 1 and 2 to 1.

30. The area of a trapezoid is $3\frac{1}{8}$ ac., the sum of the two parallel sides is 297 yds.: find the perpendicular distance between them.

31. The four sides of a quadrilateral inscribed in a circle are 80, 60, 50, and 86 ft.: required the area.

32. One of the parallel sides of a trapezoid is 1 ft. longer than the other, the breadth is 1 ft., and the area 216 sq. in.: required each of the parallels.

33. How many square yards are contained in a quadrilateral, one of its diagonals being 60 yds. and the perpendiculars upon it 12·6 and 11·4 yds.?

K. *Roorkee Upper Subordinate: Final.*

34. Find the area of a trapezoid whose parallel sides are 72 and $38\frac{2}{3}$ ft., the other sides being 20 and $26\frac{2}{3}$ ft.

35. A ditch is 30 ft. wide at top and 18 ft. at bottom. The earth excavated from it is formed into a bank 28 ft. wide at top and 38 ft. at bottom, and 10 ft. high: what is the depth of the ditch?

L. *Sandhurst.*

36. The area of a trapezoidal field is $4\frac{1}{2}$ ac., the perpendicular distance between the parallel sides is 120 yds., and one of the parallel sides is 10 ch.: find the other.

M. *Additional Examination Questions.—VII.* (For Answers, see p. 168.)

37. $ABCD$ is a quadrilateral, right-angled at B and D; also $AB = 36$ chains, $BC = 77$ chains, $CD = 68$ chains: find the area. (European Schools: Final. U.P.)

38. Find an expression for the area of a trapezoid with parallel sides of lengths a and b, and the other sides c and d. (Roorkee Engineer: Entrance.)

39. Find the area of a quadrilateral $ABCD$, given $AB = 30$ in., $BC = 17$ in., $CD = 25$ in., $DA = 28$ in., $BD = 26$ in. (Allahabad University: Matriculation.)

40. One diagonal of a quadrilateral which falls without the figure is equal to 30 yds., and the difference of the perpendiculars upon it from the remaining angles of the quadrilateral is 40 yds.: find the area. (Roorkee Upper Subordinate: Entrance.)

41. The sides of a quadrilateral are 204, 369, 325, 116 yds., and the second side is parallel to the fourth: prove that the angle contained by the first two sides is a right angle, and find the area of the quadrilateral. (European Schools: Final. U.P.)

CHAPTER VIII.

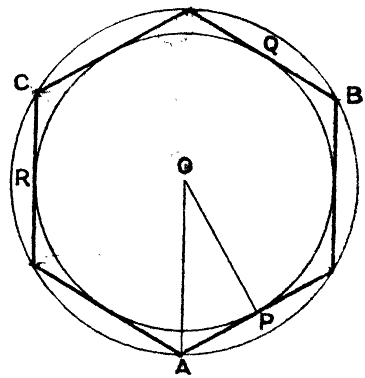
ON REGULAR POLYGONS.

43. A *polygon* is a figure bounded by four or more straight lines.

A polygon is said to be *regular* when all its sides and angles are equal.

A four-sided polygon is called			a *quadrilateral.*
A five-sided	,,	,,	a *pentagon.*
A six-sided	,,	,,	a *hexagon.*
A seven-sided	,,	,,	a *heptagon.*
An eight-sided	,,	,,	an *octagon.*
A nine-sided	,,	,,	a *nonagon.*
A ten-sided	,,	,,	a *decagon.*
An eleven-sided	,,	,,	an *undecagon.*
A twelve-sided	,,	,,	a *dodecagon.*
A fifteen-sided	,,	,,	a *quindecagon.*

It is obvious that the central point of a regular polygon is also the centre of both the circle which is circumscribed about the polygon and of the circle which is inscribed in the polygon.



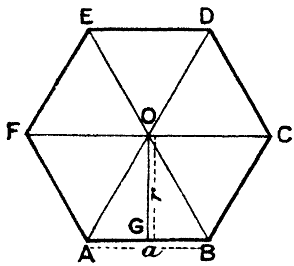
It is also evident that the perpendicular from the central point of a regular polygon upon a side is the radius of the inscribed circle, and that the straight line joining the central point to an angular point of the polygon is the radius of the circumscribed circle.

For example, O is the centre of both the circle ABC, which is circumscribed about the polygon, and of the circle PQR, which is inscribed in the polygon.

Also, OP is the radius of the circle PQR, and OA is the radius of the circle ABC.

PROPOSITION X.

44. *To find the area of a regular polygon of* n *sides, having given the length of a side and the radius of the inscribed circle.*



Let $ABCDEF$ be a regular polygon.

From its central point O draw OG perpendicular to AB.

Then OG will be the radius of the inscribed circle . . . § 43.

Let OG measure r of any linear unit. Let AB measure a of the same linear unit.

It is required to find the area of the polygon in terms of n, a, and r.

Join OA, OB, OC, OD, OE, OF.

Thus the polygon is divided up into as many equal triangles as the figure has sides.

Area of polygon = area of $\triangle AOB$ × number of sides of polygon

$= \frac{1}{2} \times OG \times AB \times$,, ,, ,, § 20.

$= \frac{n}{2} \times ar$ sq. units

Hence rule—

The product of the number of any linear unit in one side of a regular polygon by the number of the same linear unit in the radius of the inscribed circle, multiplied by half the number of the sides, gives the number of the corresponding square unit in the area.

Or briefly—

Area of a regular polygon = $\frac{\text{number of sides}}{2}$ × side × radius of inscribed circle

$$A = \frac{n}{2} \times ar \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Hence—

The side of a regular polygon

$$= \frac{2 \times \text{area}}{\text{number of sides} \times \text{radius of inscribed circle}}$$

$$a = \frac{2A}{nr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii.)$$

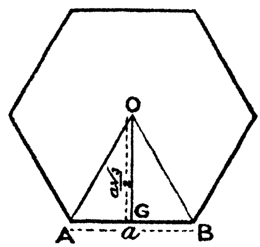
and perimeter of polygon $= na = \dfrac{2A}{r}$

$$p = \frac{2A}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii.)$$

PARTICULAR CASES.

45. 1. **Hexagon.**

Here it may be seen that AOB is an equilateral triangle.



$$\therefore OG = \frac{AB \times \sqrt{3}}{2} \quad . \quad . \quad . \quad \S\ 17.$$

that is, $r = \dfrac{a\sqrt{3}}{2}$

Now, area of any regular polygon $\Big\} = \dfrac{n}{2} \times ar$ sq. units . § 44.

$\therefore$ area of reg. hexagon $\Big\} = \dfrac{6}{2} \times a \times \dfrac{a\sqrt{3}}{2}$ sq. units

$$= \frac{3a^2\sqrt{3}}{2} \text{ sq. units}$$

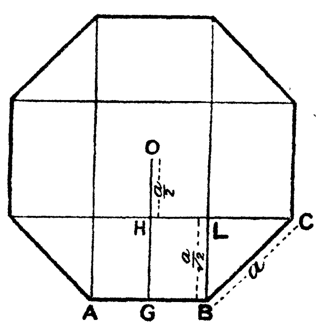
2. **Octagon.**

Here it may be seen that—

$$OG = OH + HG$$
$$= \frac{a}{2} + LB$$

But LB is the side of a square whose diagonal is $BC = a$.

$$\therefore LB = \frac{a}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 17.$$



$$\therefore OG = \frac{a}{2} + \frac{a}{\sqrt{2}}$$
$$= a\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$$
$$= a\left(\frac{1 + \sqrt{2}}{2}\right)$$

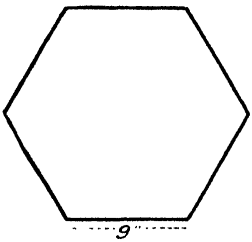
Now, area of any regular polygon $\Big\} = \dfrac{n}{2} \times ar$ sq. units § 44.

$\therefore$ area of reg. octagon $\Big\} = \dfrac{8}{2} \times a \times a\left(\dfrac{1+\sqrt{2}}{2}\right)$ sq. units

$$= 2a^2(1 + \sqrt{2}) \text{ sq.units}$$

ILLUSTRATIVE EXAMPLES.

46. *Example* 1.—Find the area of a regular hexagon whose side measures 9 in.

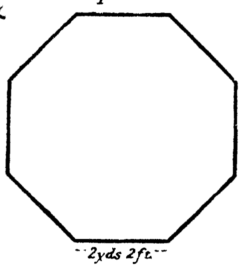


$$\left.\begin{array}{l}\text{Area of regular}\\ \quad\text{hexagon}\end{array}\right\} = \frac{3a^2\sqrt{3}}{2} \text{ sq. in.} \quad . \ \S\ 45.$$

where $a = 9$;

$$\therefore \left.\begin{array}{l}\text{area of regular}\\ \quad\text{hexagon}\end{array}\right\} = \frac{3 \times 81 \times \sqrt{3}}{2} \text{ sq. in.}$$
$$= \frac{243 \times 1{\cdot}73205\ldots}{2} \text{ sq. in.}$$
$$= 210{\cdot}44\ldots \text{ sq. in.}$$

* *Example* 2.—Find the area of a regular octagon whose side measures 2 yds. 2 ft.

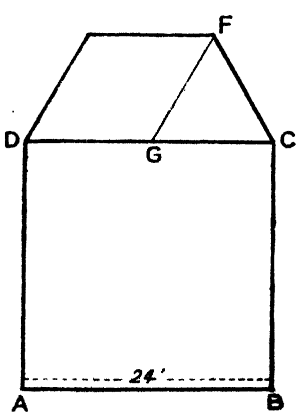


$$\left.\begin{array}{l}\text{Area of regular}\\ \quad\text{octagon}\end{array}\right\} = 2a^2(1+\sqrt{2}) \text{ sq. ft.} \quad . \ \S\ 45.$$

$$\text{where } a = (2 \times 3 + 2)$$
$$= 8;$$

$$\therefore \text{ area of octagon} = 2 \times 8^2(1+\sqrt{2}) \text{ sq. ft.}$$
$$= 128(1+\sqrt{2}) \text{ sq. ft.}$$
$$= 128 \times 2{\cdot}41421\ldots \text{ sq. ft.}$$
$$= 309{\cdot}01\ldots \text{ sq. ft.}$$
$$= 34 \text{ sq. yds. } 3{\cdot}01\ldots \text{ sq. ft.}$$

Example 3.—There is a square room which it is proposed to enlarge by throwing out a hexagonal front on one of its sides, so that three sides of the hexagon may form a bay front: what area of new flooring will be required, the side of the square being 24 ft.?



In the figure (which represents a plan of the floor), join G, the middle point of the side DC of the square $ABCD$, to F.

Then it is evident that FGC is an equilateral triangle.

$$\therefore FC = GC = \tfrac{1}{2} . DC = 12 \text{ ft.}$$

Hence area of new flooring will equal area of half a regular hexagon whose side measures 12 ft.

$$= \tfrac{1}{2} . \frac{3a^2\sqrt{3}}{2} \text{ sq. ft.} \quad . \ \S\ 45.$$

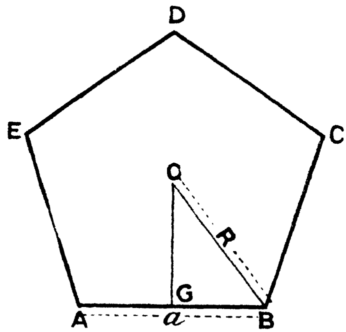
where $a = 12$;

Hence—

$$\text{area of new flooring} = \frac{3 \times 144 \times \sqrt{3}}{4} \text{ sq. ft.}$$
$$= 187{\cdot}061 \ldots \text{ sq. ft.}$$

PROPOSITION XI.

47. *To find the area of a regular polygon of* n *sides, having given the length of a side and the radius of the circumscribed circle.*



Let $ABCDE$ be a regular polygon.

Join O, its central point, to B.

Then OB will be the radius of the circumscribed circle.

Let OB measure R of any linear unit. Let AB measure a of the same linear unit.

It is required to find the area of the polygon in terms of n, a, and R.

Draw OG perpendicular to AB.

Now $\because$ OG is the radius of the inscribed circle . . § 43.

$$\therefore \text{ area of polygon} = \frac{n}{2} \times a \times OG \quad \ldots \text{ § 44.}$$

$$\text{But } OG = \sqrt{OB^2 - GB^2} \quad \ldots \text{ § 16.}$$
$$= \sqrt{R^2 - \left(\frac{a}{2}\right)^2}$$

$$\therefore \text{ area of polygon} = \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2} \text{ sq. units}$$

$$\text{and } \because na = \text{perimeter of polygon}$$

$$\therefore \text{ area of polygon} = \frac{\text{perimeter}}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$$
$$\text{or } \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2} \text{ sq. units}$$

Hence rule—

From the square of the number of any linear unit in the radius of the circumscribed circle subtract the square of half the number of the same linear unit in a side of the polygon; then the square root of the remainder, multiplied by half the number of the same linear unit in the perimeter, gives the number of the corresponding square unit in the area.

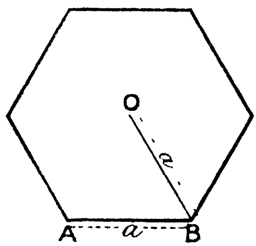
Or briefly—

Area of regular polygon

$$= \frac{1}{2} \cdot \text{perimeter}\sqrt{(\text{radius of circumd. circle})^2 - \left(\frac{\text{side}}{2}\right)^2}$$

$$A = \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$$

PARTICULAR CASES.



48. 1. **Hexagon.**

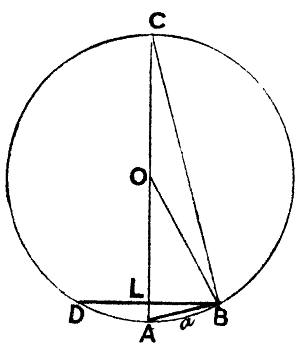
Here $OB = R = AB = a$, and area of any reg. polygon $= \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$ sq. units . § 47.

$\therefore$ area of regular hexagon $= \frac{6a}{2}\sqrt{a^2 - \frac{a^2}{4}}$ sq.units

$= \frac{3a^2\sqrt{3}}{2}$ sq. units

This result has been previously obtained in § 45.

2. **Dodecagon.**



Let AB be a side of a regular dodecagon, and BD a side of a regular hexagon, inscribed in the circle $ABCD$.

Through O, the centre of the circle, draw COL perpendicular to DB.

Then COL produced will pass through A.

Join OB and CB.

Then OB is the radius of the circumscribed circle . . . § 43.

That is, $OB = R = 2 . LB$

Now, because the triangles ABC and ABL are similar,

$\therefore CA : AB = AB : AL$ . . . Euc. VI. 4.

i.e. $2R : a = a : AL$

But $AL^2 = AB^2 - BL^2$. Euc. I. 47.

$$= a^2 - \left(\frac{R}{2}\right)^2$$

$$\therefore 2R : a = a : \sqrt{a^2 - \frac{R^2}{4}}$$

$$\therefore 2R\sqrt{a^2 - \frac{R^2}{4}} = a^2$$

$$\therefore R^2 = a^2(2 + \sqrt{3})$$

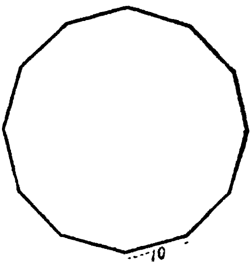
Now, area of any reg. polygon $= \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$ sq. units § 47.

$\therefore$ " " dodecagon $= \frac{12a}{2}\sqrt{a^2(2+\sqrt{3}) - \frac{a^2}{4}}$ sq. units

$= 6a^2\sqrt{\frac{7}{4}+\sqrt{3}}$ sq. units

ILLUSTRATIVE EXAMPLES.

49. *Example* 1.—Find the area of a regular dodecagon, whose side measures 10 in.



Area of regular dodecagon $= 6a^2\sqrt{\frac{7}{4}+\sqrt{3}}$ sq. in. § 48.

where $a = 10$;

$\therefore$ area $= 600\sqrt{\frac{7}{4}+\sqrt{3}}$ sq. in.
$= 600\sqrt{1\cdot75 + 1\cdot7320508\ldots}$ sq. in.
$= 600\sqrt{3\cdot4820508\ldots}$ sq. in.
$= 600 \times 1\cdot8660$ sq. in. nearly
$= 1119$ sq. in. nearly

Example 2.—Find the side of a regular hexagonal enclosure which measures one acre.

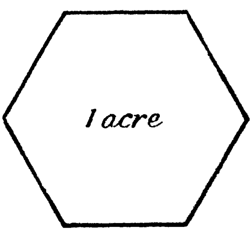
Area of regular hexagon $= \frac{3a^2\sqrt{3}}{2}$ sq. yds. . . § 45.

$\therefore \frac{3a^2\sqrt{3}}{2}$ sq. yds. $= 1$ acre $= 4840$ sq. yds.

$\therefore a^2 = \frac{2 \times 4840}{3\sqrt{3}}$

$\therefore a = \sqrt{\frac{2 \times 4840}{3\sqrt{3}}}$

$\therefore$ side of reg. hexagon $= \sqrt{\frac{2 \times 4840 \times \sqrt{3}}{3\sqrt{3} \times \sqrt{3}}}$ yds.
$= \sqrt{\frac{9680 \times \sqrt{3}}{9}}$ yds.
$= \sqrt{\frac{9680 \times 1\cdot73205}{9}}$ yds. nrly.
$= \sqrt{\frac{16766}{9}}$ yds. nearly
$= \frac{129}{3}$ yds. nearly
$= 43$ yds. nearly



* *Example* 3.—Compare the areas of a regular octagon and a regular dodecagon of equal perimeter.

Let the perimeter of each measure x in.

Then each side of the octagon measures $\frac{x}{8}$ in., and each side of the dodecagon measures $\frac{x}{12}$ in.

But area of octagon $= 2a^2(1 + \sqrt{2})$ sq. in. . . § 45.

where $a = \frac{x}{8}$;

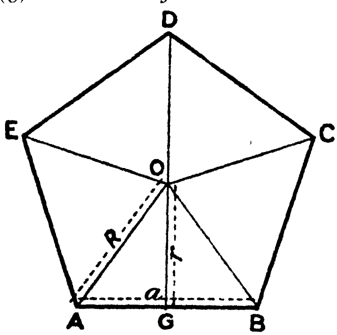
and area of dodecagon $= 6a^2\sqrt{\tfrac{7}{4} + \sqrt{3}}$ sq. in. . . § 48.

where $a = \frac{x}{12}$;

$$\therefore \text{area of octagon} : \text{area of dodecagon} = \frac{2x^2(1+\sqrt{2})}{64} : \frac{6x^2\sqrt{\tfrac{7}{4}+\sqrt{3}}}{144}$$
$$= \frac{1+\sqrt{2}}{32} : \frac{\sqrt{\tfrac{7}{4}+\sqrt{3}}}{24}$$
$$= 3(1+\sqrt{2}) : 4\sqrt{\tfrac{7}{4}+\sqrt{3}}$$

PROPOSITION XII.

50. *To find the area of a regular polygon of* n *sides, having given* (1) *the length of a side, or* (2) *the radius of the inscribed circle, or* (3) *the radius of the circumscribed circle.*



Let $ABCDE$ be a regular polygon.

Join its central point O to its angular points A, B, C, D, E. Draw OG perpendicular to AB.

Then OA is the radius of the circumscribed circle, and OG is the radius of the inscribed circle.

Let AB measure a of any linear unit. Let OG measure r of the same linear unit. Let OA measure R of the same linear unit

It is required to find the area of the polygon in terms of—

(1) n and a,
(2) n and r,
(3) n and R.

Now, area of polygon $= \frac{n}{2} . a \times OG$ . . . . § 44.

$$\text{and } OG = AG \cot AOG$$
$$= \frac{a}{2} \cot \frac{180^\circ}{n}$$

$$\therefore \text{ area of polygon} = \frac{n}{2} \times a \times \frac{a}{2} \cot \frac{180^\circ}{n} \text{ sq. units}$$

$$= a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n} \text{ sq. units} \quad . \quad . \ (A)$$

$$\text{Again, } \because \text{ area of polygon} = \frac{n}{2} . AB \times r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ \S\ 44.$$

$$\text{and } AB = 2 . OG \tan AOG$$

$$= 2r \tan \frac{180^\circ}{n}$$

$$\therefore \text{ area of polygon} = \frac{n}{2} . 2r \tan \frac{180^\circ}{n} . r \text{ sq. units}$$

$$= r^2 \times n \tan \frac{180^\circ}{n} \text{ sq. units} \quad . \quad . \ (B)$$

$$\text{Lastly, } \because \text{ area of polygon} = \triangle AOB \times n$$

$$\text{and } \triangle AOB = \tfrac{1}{2} . OA . OB \sin AOB$$

$$= \tfrac{1}{2} . R^2 \sin \frac{360^\circ}{n}$$

$$\therefore \text{ area of polygon} = \tfrac{1}{2} . R^2 \sin \frac{360^\circ}{n} \times n \text{ sq. units}$$

$$= R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n} \text{ sq. units} \quad . \quad . \ (C)$$

Hence rule—

The number of any square unit in a regular polygon of n *sides is obtained by multiplying—*

(A) *the square of the number of the corresponding linear unit in a side by* $\frac{n}{4}$ *cot* $\frac{180^\circ}{n}$;

(B) *the square of the number of the corresponding linear unit in the radius of the inscribed circle by* n *tan* $\frac{180^\circ}{n}$;

(C) *the square of the number of the corresponding linear unit in the radius of the circumscribed circle by* $\frac{n}{2}$ *sin* $\frac{360^\circ}{n}$.

Or briefly—

(1) **Area of polygon of n sides** $= (\text{side})^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}$

$$= a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}$$

(2) „ „ „ $= (\text{radius of inscrbd. circle})^2 \times n \tan \frac{180^\circ}{n}$

$$= r^2 \times n \tan \frac{180^\circ}{n}$$

(3) **Area of polygon of n sides = (radius of circumd. circle)^2**

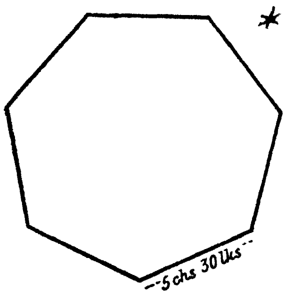
$$\times \frac{n}{2} \sin \frac{360^\circ}{n}$$

$$= R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n}$$

The following table gives the values correct to four places of decimals of the multipliers $\frac{n}{4} \cot \frac{180^\circ}{n}$, $n \tan \frac{180^\circ}{n}$, $\frac{n}{2} \sin \frac{360^\circ}{n}$ for some polygons of frequent occurrence :—

Name of polygon.	$\frac{n}{4} \cot \frac{180^\circ}{n}$	$n \tan \frac{180^\circ}{n}$	$\frac{n}{2} \sin \frac{360^\circ}{n}$
Pentagon . .	1·7204	3·6327	2·3776
Hexagon .	2·5980	3·4641	2·5980
Heptagon . .	3·6339	3·3710	2·7364
Octagon . . .	4·8284	3·3137	2·8284
Nonagon .	6·1818	3·2757	2·8925
Decagon . .	7·6942	3·2492	2·9389
Undecagon .	9·3656	3·2299	2·9735
Dodecagon . .	11·1961	3·2153	3·0000

ILLUSTRATIVE EXAMPLES.



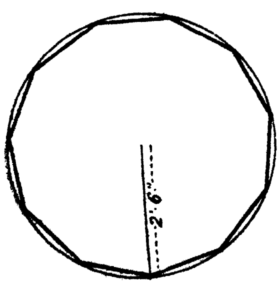
51. *Example* 1.—Find the area of a regular heptagon whose side measures 5 ch. 30 lks.

$\left.\begin{array}{r}\text{Area of reg.}\\ \text{heptagon}\end{array}\right\} = a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}$ sq. ch. § 50.

where $a = 5.3$,

and $\frac{n}{4} \cot \frac{180^\circ}{n} = 3·6339$; . . . § 50.

$\therefore$ area $= (5·3)^2 \times 3·6339$ sq. ch.
$= 102·0762$ sq. ch. nearly
$= 10$ ac. 2 sq. ch. 762 sq. lks. nearly



Example 2.—A regular undecagon is inscribed in a circle whose radius is 2 ft. 6 in. : find the area of the undecagon.

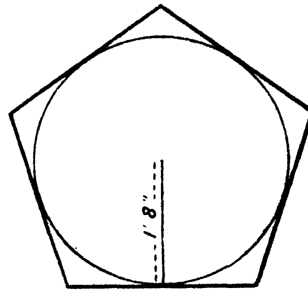
$\left.\begin{array}{r}\text{Area of reg.}\\ \text{undecagon}\end{array}\right\} = R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n}$ sq. in. § 50.

where $R = (2 \times 12 + 6) = 30$,

and $\frac{n}{2} \sin \frac{360^\circ}{n} = 2·9735$; . . . § 50.

$\therefore$ area $= (30)^2 \times 2·9735$ sq. in.
$= 2676·1$ sq. in. nearly
$= 18$ sq. ft. 84·1 sq. in. nearly

Example 3.—A regular pentagon is circumscribed about a circle whose radius is 1 ft. 8 in.: find the area of the pentagon.



$$\left.\begin{array}{r}\text{Area of}\\ \text{pentagon}\end{array}\right\}=r^2\times n\tan\frac{180^\circ}{n}\text{ sq. in. § 50.}$$

where $r = 1 \times 12 + 8 = 20$,

and $n \tan \frac{180^\circ}{n} = 3{\cdot}6327$; . . § 50.

$$\therefore \left.\begin{array}{r}\text{area of}\\ \text{pentagon}\end{array}\right\}=(20)^2 \times 3{\cdot}6327 \text{ sq. in.}$$

$= 1453$ sq. in. nearly

$= 10$ sq. ft. 13 sq. in. nearly

Examples—VIII. A.

1. Find the area of a regular pentagon whose side measures 5 ch. 55 lks., and the radius of the inscribed circle is 3 ch. 82 lks.

2. Find the area of a regular hexagon whose side measures 8 in.

3. Find the area of a regular hexagon whose side measures 3 ch. 25 lks. (give the result in acres).

4. Find the area of a regular hexagon inscribed in a circle whose radius is 15 ft.

5. Find the area of a regular hexagon described about a circle whose radius is $7\sqrt{3}$ in.

6. Find the area of a regular nonagon whose side measures 10 lks.

7. Find the area of a regular pentagon whose side measures 6 yds.

8. Find the cost of carpeting an octagonal floor whose side measures 16 ft., at Rs.2 per square yard ($\sqrt{2} = 1{\cdot}41421$).

9. It costs Rs.900 to fence a dodecagonal enclosure at Rs.5 a yard: find its area.

Examples—VIII. B.

10. Find the area of a regular hexagon whose side measures half a rasi (give the result in biswas).

11. Find the area of a regular octagon whose side measures 2 rasi (give the result in bighas).

12. What must be the side of a regular hexagon in order that it may just enclose 1 bigha? (give the result in rasi).

Examination Questions—VIII.

A. *Punjab University: Matriculation.*

1. An ornamental grass-plot is in the shape of a regular hexagon, each side 100 ft.; within the plot and along its sides a footpath is made, 4 ft. wide all round: find the area of the grass-plot left within.

B. *Calcutta University: Matriculation.*

2. Calculate to three decimal places the area of a regular hexagon, each of whose sides is equal to 10 ft.

3. Find the area of a regular octagonal field each of whose sides measures 5 ch. (give the result in acres, roods, etc.).

C. *Roorkee Engineer: Entrance.*

4. There is a square room which it is proposed to enlarge by throwing out

an octagonal front on one of its sides, so that three sides of the octagon may form a bay front; what area of new flooring will be required, the side of the square being 20 ft.?

5. A regular decagon is inscribed in a circle, the radius of which is 10 in.: find the area of the polygon.

6. Compare the areas of an equilateral triangle, a square, and a regular hexagon of equal perimeter.

7. The area of a regular octagon is 51 sq. yds.: find the length of its side.

8. The radius of the circumscribed circle of a pentagon is $\sqrt{\frac{3000}{\pi}}$ ft., where $\pi = 3 \cdot 1416$: find the length of the side, and the area of the pentagon.

D. *Roorkee Engineer: Final.*

9. Find the side of a regular octagon inscribed in a square, the area of which is $6 + 4\sqrt{2}$ sq. ft.

10. Find the area of a regular heptagon whose side is 45 ft.

11. If a regular hexagon, a square, and an equilateral triangle be inscribed in a circle of 12 ft. diameter, show that the square described upon the side of the triangle is equal to the sum of the squares described upon one side of each of the other two figures.

12. The side of a regular pentagon inscribed in a circle is 1 ft.: find the radius of the circle.

13. Deduce a formula for the area of a regular polygon in terms of the number and length of its sides, and from this find the area of a regular heptagon, the length of each side being 2 ft.

E. *Roorkee Upper Subordinate: Monthly.*

14. The radius of a circle is 1 ft.: find the area of a regular polygon of eight sides inscribed in the circle.

15. Find the area of a regular octagon inscribed in a square, the side of which is $10\sqrt{2}$.

16. A regular dodecagon is inscribed in a circle, of which the radius is 3 in.: find the area of the polygon in feet.

17. Find the side of a regular octagon whose area is 1 ro., and of a regular dodecagon whose area is 1000 yds.

18. Find the area of a regular hexagon whose perimeter is 3000 ft.

19. Find the area of a hexagon, each side being 30 ft.

20. Find the area of a regular octagon whose side is 20 ft.

21. The radius of a circle is 1 ft.: find the area of a regular dodecagon inscribed in the circle.

22. The front of a room 24 ft. wide is to be projected in the form of three sides of an octagon: construct the projection. How much will it increase the total length of the room in the central line, and how much will it add to the area?

23. The area of a regular octagon is $1086 \cdot 4$ ft.: find the length of one side.

24. Find the areas of a square and a regular hexagon, the perimeter of each being 300 ft.

F. *Additional Examination Questions.—VIII.* (For Answers, see p. 168.)

25. Find the area of a polygon of twenty-five sides inscribed in a circle of radius 10 ft. Sin $14^\circ 24' = 0 \cdot 249$. (Roorkee Engineer: Entrance.)

26. The radius of a circle is 12 ft. Find the length of the side of a polygon of sixteen sides inscribed in it. Calculations to be made to three places of decimals. (Punjab University: First Examination in Civil Engineering.)

CHAPTER IX.

ON IRREGULAR RECTILINEAL FIGURES.

52. Consider the irregular rectilineal figure *ABCDEFG* (Fig. 1).

If we can divide it up into such parts that the area of each part may be ascertained separately, then we shall be able to find the area of the whole figure by adding together the areas of the parts. The division into such parts may be done by means of *base-lines* (or chain lines) and *offsets*.

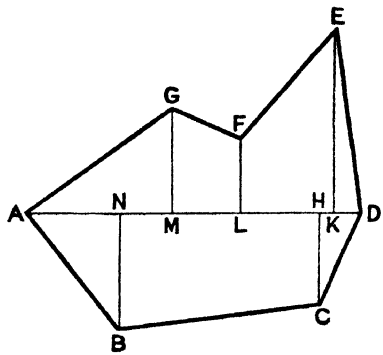


Fig. 1.

53. A *base-line* is a straight line drawn from one angular point to another, and *offsets* are perpendiculars drawn from the other angular points to a base-line.

One base-line will often suffice, but it is sometimes convenient to draw two or more base-lines.

The figure *ABCDEFG* (Fig. 1) may be conveniently divided into parts by drawing the base-line *AD*, and the offsets *BN*, *CH*, *EK*, *FL*, *GM*. These parts are either triangles or trapezoids, and their areas may be ascertained if we know the lengths of—

(*a*) the offsets;

(*b*) the segments of the base-line made by the offsets.

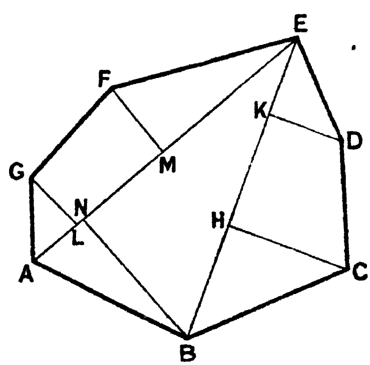


Fig. 2.

54. Again, consider the irregular rectilineal figure *ABCDEFG* (Fig. 2).

This may be conveniently divided into parts by means of the base-lines *AE* and *BE*, and the offsets *GL*, *FM*, *DK*, *CH*, *BN*.

55. Sometimes it is found convenient to draw base-lines which may wholly or partially lie *outside* the figure.

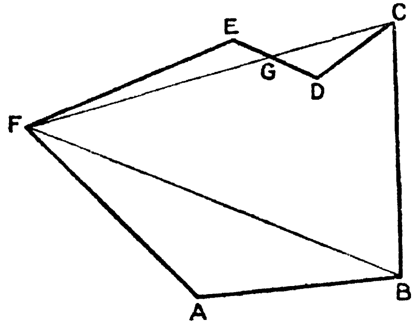


FIG. 3.

For example, consider the irregular rectilineal figure *ABCDEF* (Fig. 3).

If we draw the base-lines *FC* and *FB*, the area of the figure may be regarded as the sum of the areas of the $\triangle FBC$, the $\triangle ABF$, and the $\triangle FEG$, *less* the area of the $\triangle GDC$.

When the perpendicular from an angular point upon a base-line lies outside the figure, it is called an *inset*.

Thus we speak of the *inset* from the base-line *FC* to the angular point *D*.

56. Again, consider the irregular rectilineal figure *ABCDEF* (Fig. 4).

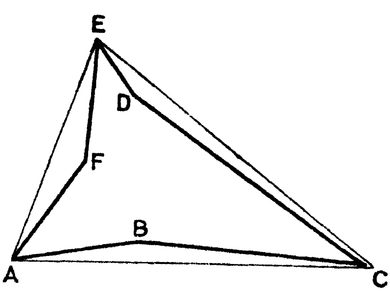


FIG. 4.

If we draw the base-lines *AC*, *CE*, *EA*, then the area of the figure may be regarded as equal to the area of the $\triangle ACE$, less the sum of the areas of the triangles *ABC*, *CDE*, and *EAF*.

This method practically amounts to—

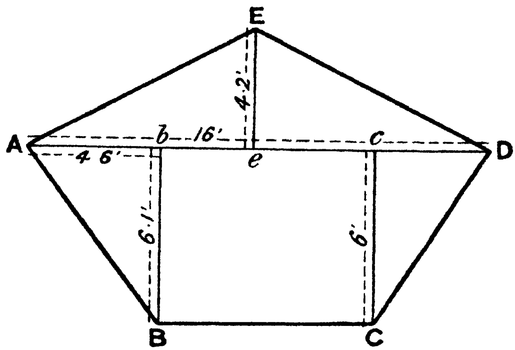
(1) Finding the area of the figure bounded by the base-lines.

(2) Adding the areas of those figures which lie outside the figure bounded by the base-lines.

(3) Subtracting the areas of those figures which lie inside the figure bounded by the base-lines.

ILLUSTRATIVE EXAMPLES.

57. *Example* 1.—$ABCDE$ is a five-sided rectilineal figure; Bb, Cc, Ee are perpendiculars from B, C, E on AD respectively.



It is given that $AD = 16$ ft.
$Ee = 4{\cdot}2$ ft.
$Bb = 6{\cdot}1$ ft.
$Ab = 4{\cdot}6$ ft.
$Ac = 12$ ft.
$Cc = 6$ ft.

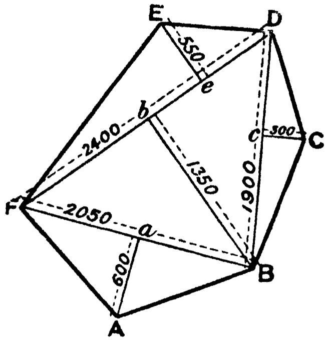
Find the area.

$$\text{Because } bc = Ac - Ab$$
$$\therefore bc = (12 - 4{\cdot}6)\text{ ft.} = 7{\cdot}4 \text{ ft.}$$
$$\text{and because } cD = AD - Ac$$
$$\therefore cD = (16 - 12)\text{ ft.} = 4 \text{ ft.}$$

Area of figure $ABCDE$ = area of $\triangle AED$ + area of $\triangle ABb$ + area of $\triangle CDc$ + area of trapezoid $BCcb$.

$$\begin{aligned}
\text{Area of } \triangle AED &= \tfrac{1}{2} \times AD \times Ee \quad \ldots\ldots \S\ 20.\\
&= \tfrac{1}{2} \times 16 \times 4{\cdot}2 \text{ sq. ft.}\\
&= 33{\cdot}6 \text{ sq. ft.}\\
\text{area of } \triangle ABb &= \tfrac{1}{2} \times Ab \times Bb \quad \ldots\ldots \S\ 20.\\
&= \tfrac{1}{2} \times 4{\cdot}6 \times 6{\cdot}1 \text{ sq. ft.}\\
&= 14{\cdot}03 \text{ sq. ft.}\\
\text{area of } \triangle CDc &= \tfrac{1}{2} \times cD \times Cc \quad \ldots\ldots \S\ 20.\\
&= \tfrac{1}{2} \times 4 \times 6 \text{ sq. ft.}\\
&= 12 \text{ sq. ft.}\\
\text{area of trapezoid } BCcb &= \tfrac{1}{2} \times bc \times (Bb + Cc) \quad \ldots\ldots \S\ 39.\\
&= \tfrac{1}{2} \times 7{\cdot}4 \times 12{\cdot}1 \text{ sq. ft.}\\
&= 44{\cdot}77 \text{ sq. ft.}\\
\therefore \text{area of figure } ABCDE &= (33{\cdot}6 + 14{\cdot}03 + 12 + 44{\cdot}77) \text{ sq. ft.}\\
&= 104{\cdot}4 \text{ sq. ft.}
\end{aligned}$$

Example 2.—Find in acres the area of the figure $ABCDEF$, having given—



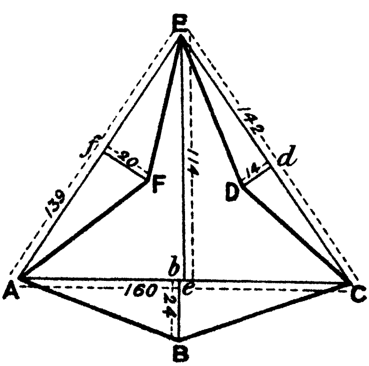
FD 2400 lks.
FB 2050 „
BD 1900 „
Bb 1350 „
Aa 600 „
Cc 300 „
Ee 550 „

Area of figure $ABCDEF$ = area of $\triangle FBD$ + area of $\triangle FAB$ + area of $\triangle BCD$ + area of $\triangle FDE$.

Now, area of $\triangle FBD$ $= \frac{1}{2} \times FD \times Bb$ . . . § 20.
$= \frac{1}{2} \times 2400 \times 1350$ sq. lks.
$= 1{,}620{,}000$ sq. lks.

area of $\triangle FAB = \frac{1}{2} \times FB \times Aa$ . . . . § 20.
$= \frac{1}{2} \times 2050 \times 600$ sq. lks.
$= 615{,}000$ sq. lks.
area of $\triangle BCD = \frac{1}{2} \times BD \times Cc$ . . . . § 20.
$= \frac{1}{2} \times 1900 \times 300$ sq. lks.
$= 285{,}000$ sq. lks.
area of $\triangle FDE = \frac{1}{2} \times FD \times Ee$ . . . . § 20.
$= \frac{1}{2} \times 2400 \times 550$ sq. lks.
$= 660{,}000$ sq. lks.
∴ area of figure $= (1{,}620{,}000 + 615{,}000 + 285{,}000 + 660{,}000)$ sq. lks.
$= 3{,}180{,}000$ sq. lks.
$= 31 \cdot 8$ ac.

Example 3.—Find in square yards, square feet, and square inches the area of the rectilineal figure $ABCDEF$, having given—



$AC = 160$ in.
$CE = 142$ „
$EA = 139$ „
$Bb = 24$ „
$Dd = 14$ „
$Ff = 20$ „
$Ee = 114$ „

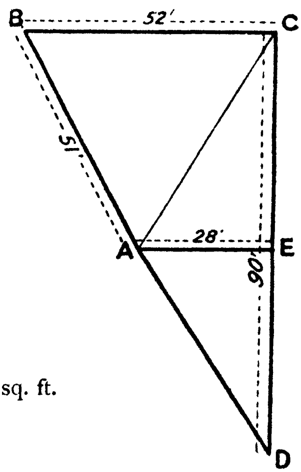
Area of figure $ABCDEF$ = area of $\triangle ACE$ + area of $\triangle ABC$ − area of $\triangle CDE$ − area of $\triangle EAF$.

Now, area of $\triangle ACE$ $= \frac{1}{2} AC \times Ee$. § 20.
$= \frac{1}{2} \times 160 \times 114$ sq. in.
$= 9120$ sq. in.

area of $\triangle ABC = \frac{1}{2} \times AC \times Bb$ . . . . § 20.
$= \frac{1}{2} \times 160 \times 24$ sq. in.
$= 1920$ sq. in.

$$\begin{aligned}
\text{area of } \triangle CDE &= \tfrac{1}{2} \times CE \times Dd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 20. \\
&= \tfrac{1}{2} \times 142 \times 14 \text{ sq. in.} \\
&= 994 \text{ sq. in.} \\
\text{area of } \triangle EAF &= \tfrac{1}{2} \times EA \times Ff \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 20. \\
&= \tfrac{1}{2} \times 139 \times 20 \text{ sq. in.} \\
&= 1390 \text{ sq. in.} \\
\therefore \text{ area of figure} &= (9120 + 1920 - 994 - 1390) \text{ sq. in.} \\
&= (11{,}040 - 2384) \text{ sq. in.} \\
&= 8656 \text{ sq. in.} \\
&= 6 \text{ sq. yds. } 6 \text{ sq. ft. } 16 \text{ sq. in.}
\end{aligned}$$

Example 4.—$ABCD$ is a quadrilateral in which $AB = 51$ ft., $BC = 52$ ft., $CD = 90$ ft., the perpendicular AE from A on DC is 28 ft., and it divides DC into two equal parts : find the area of $ABCD$



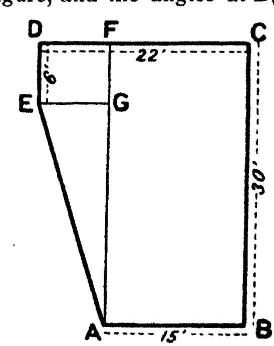
Area of $ABCD$ = area of $\triangle ACD$ + area of $\triangle ABC$

$$\begin{aligned}
\left.\begin{array}{r}\text{Now, area}\\ \text{of } \triangle ACD\end{array}\right\} &= \tfrac{1}{2} \,.\, AE \,.\, CD \quad . \quad . \quad . \quad \S\ 20. \\
&= \tfrac{1}{2} \times 28 \times 90 \text{ sq. ft.} \\
&= 1260 \text{ sq. ft.} \\
\left.\begin{array}{r}\text{and area}\\ \text{of } \triangle ABC\end{array}\right\} &= \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. ft.} \quad \S\ 23.
\end{aligned}$$

$$\begin{aligned}
\text{where } a &= 51, \\
b &= 52, \\
c &= \sqrt{(45)^2 + (28)^2} = 53 \quad . \quad . \quad \S\ 16. \\
\therefore s &= 78
\end{aligned}$$

$$\begin{aligned}
\left.\begin{array}{r}\therefore \text{ area of}\\ \triangle ABC\end{array}\right\} &= \sqrt{78 \times 27 \times 26 \times 25} \text{ sq. ft.} \\
&= \sqrt{2^2 \times 13^2 \times 3^2 \times 5^2 \times 3^2} \text{ sq. ft.} \\
&= 1170 \text{ sq. ft.} \\
\therefore \text{ area of } ABCD &= (1260 + 1170) \text{ sq. ft.} \\
&= 2430 \text{ sq. ft.}
\end{aligned}$$

Example 5.—$ABCDE$ is a five-sided figure, and the angles at B, C, and D are right angles. If $AB = 15$ ft., $BC = 30$ ft., $CD = 22$ ft., and $DE = 6$ ft., find the area of the figure and the length of EA.



Through A draw AF parallel to BC, and through E draw EG parallel to DF.

$$\begin{aligned}
\left.\begin{array}{r}\text{Now, area}\\ \text{of } ABCDE\end{array}\right\} &= \text{area of } ABCF + \text{area of } DEGF + \text{area of } AGE \\
&= AB \times BC + DE \times EG + \tfrac{1}{2} \times EG \times AG \qquad \S\S\ 8,\ 21. \\
&= 15 \times 30 \text{ sq. ft.} + 6 \quad 7 \text{ sq. ft.} + \tfrac{1}{2} \times 7 \times 24 \text{ sq. ft.} \\
&= (450 + 42 + 84) \text{ sq. ft.} \\
&= 576 \text{ sq. ft.}
\end{aligned}$$

Again—

$$\begin{aligned} EA &= \sqrt{EG^2 + GA^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ \S\ 16. \\ &= \sqrt{(7)^2 + (24)^2} \text{ ft.} \\ &= 25 \text{ ft.} \end{aligned}$$

Examples—IX. A.

1. Find the area in acres of the quadrilateral $ABCD$, having given $AC =$ 6000 lks., and the perpendiculars from B and D on AC 4800 lks. and 1800 lks.

2. $ABCDE$ is a five-sided figure, in which AD measures 12 ft., AC 11 ft. 6 in., the perpendiculars from C and E on AD 6 ft. and 2 ft. 6 in. respectively, and the perpendicular from B on AC 2 ft. 9 in.: find the area.

3. Find the area in square inches of a five-sided figure $ABCDE$, in which AC measures 16 in., AD $14\frac{1}{2}$ in., the perpendicular from B on AC 6 in., the perpendicular from D on AC 8 in., and the perpendicular from E on AD $4\frac{3}{4}$ in.

4. In the five-sided figure $ABCDE$ the angle at A is a right angle, CE measures 24 ft., AB 6·5 ft., AE 4·4 ft., the perpendicular from B on CE 4·8 ft., the perpendicular from D on CE 2·3 ft.: find the area.

5. In the five-sided figure $ABCDE$ the angle at A is a right angle, and DE is parallel to AB; also AB measures 1600 lks., BD 1060 lks., DE 900 lks., EA 770 lks., and the perpendicular from C on BD 430 lks.: find the area.

Examples—IX. B.

6. How many bighas are there in the quadrilateral $ABCD$, if AC measures 5·6 rasi, and if the perpendiculars from B and D on AC measure 3·4 and 3·8 rasi respectively?

7. Find the area in bighas and biswas of a five-sided figure $ABCDE$, in which AD measures 4·8 rasi, AC 3·8 rasi, the perpendiculars from C and E on AD 1·4 and 1·9 rasi respectively, and the perpendicular from B on AC 2 rasi.

8. In the five-sided figure $ABCDE$ the angles at A and D are right angles; and $AB = 7·4$ rasi, $CD = 6$ rasi, $DE = 3$ rasi, $EA = 4·6$ rasi, and the perpendicular from C on $EB = 2·4$ rasi: find the area of the figure in bighas.

Examination Questions—IX.

A. *Allahabad University: Matriculation.*

1. Make a rough sketch, and find the area of a field $ABCD$ from the following measures taken in links, and find the length of the perpendicular from A on CD:—

BM, the perpendicular from B on AC, $= 400$
DN, ,, ,, ,, D on AC, $= 300$
$AM = 300$ $AN = 400$ $AC = 625$

2. In a quadrilateral figure $ABCD$, $AB = BC = CD = 60$ yds.; $AD =$ 80 yds., and the angle DAB is a right angle: find the area of the figure.

3. The sides of a five-sided figure $ABCDE$ are $AB = 25$ ft., $BC = 29$ ft., $CD = 39$ ft., $DE = 42$ ft., and $EA = 27$ ft.; also $AC = 36$ ft., and $CE = 45$ ft.: find its area.

B. *Punjab University: Matriculation.*

4. $ABCD$ is a quadrilateral, in which $AB = 13$ ft., $BC = 14$ ft., $CD =$ 24 ft., the perpendicular from A on DC is 9 ft., and it divides DC into two equal parts: find the area of $ABCD$.

C. *Roorkee Engineer: Entrance.*

5. $ABCDE$ is a five-sided figure, and the angles at B, C, and D are right angles: if $AB = 20$ ft., $BC = 18$ ft., $CD = 32$ ft., and $DE = 13$ ft., find the area of the figure, and the length of AE.

6. The sides of a pentagon taken in order are 100, 130, 197, 133, and 94 ft., and the two diagonals measured from the intersection of the first and last sides are 209 and 193 ft.: find the area of the figure.

D. *Sandhurst.*

7. In the pentagonal field $ABCDE$, the length of AC is 50 yds., and the perpendiculars from B, D, and E upon AC are 10, 20, and 15 yds., the distances from A of the feet of the perpendiculars from D and E being 40 and 10 yds.: find the area.

E. *Militia: Literary.*

8. The lengths of the sides (in yards) of a six-sided field $ABCDEF$ are as follows: $AB = 31$, $BC = 130$, $CD = 38$, $DE = 41$, $EF = 130$, $FA = 22$. Given that the angles at A and D are right angles, and that BF is parallel to CE, find the area of the field in square yards.

F. *Additional Examination Question.—IX.* (For Answers, see p. 168.)

9. What is the content of the eight-sided figure $ABCDEFGH$, the diagonal AE being taken as a base, and the perpendiculars drawn from the angular points to AE being Bb, Cc, Dd, etc.; and when the lengths of the perpendiculars above the diagonal are $Bb = 294$, $Cc = 142{\cdot}5$, $Dd = 224$; and those below the diagonal are $Ff = 121$, $Gg = 105{\cdot}5$, $Hh = 142$; and the intercepted breadths are $Ah = 44{\cdot}5$, $hb = 124{\cdot}25$, $bg = 80$, $gc = 41$, $cd = 130{\cdot}5$, $df = 50$, $fE = 52{\cdot}5$? (Punjab University: Matriculation.)

CHAPTER X.

ON THE FIELD-BOOK.

58. The area of a field can be ascertained by means of base-lines and offsets, when its boundary may be regarded as a rectilineal figure.

59. Distances along the base-lines and offsets are measured with an instrument which is called *Gunter's chain*. This chain is 22 yds. long, and consists of 100 links.

60. The surveyor enters the record of these measurements in his *Field-book*.

Each page of the field-book is divided into three columns. The central column contains the measurements made along the base-lines; the side columns contain the measurements made along the offsets.

The ends of a base-line are called *stations*.

61. The surveyor begins his entries at the bottom of the central column, and writes upwards.

The first entry tells the direction of the first base-line. The second entry tells the distance along this base-line from the first station to the first offset. The third entry tells the length of this offset, and appears in the right- or left-hand column, according as the offset springs to the right or left of the base-line. The fourth entry tells the distance along the base-line from the first station to the second offset. The fifth entry tells the length of this offset. And so on until he reaches the end of the first base-line, which is called the "second station."

If a single base-line has been used in surveying the field, no further measurements will be required. If two or more base-lines have been used, the surveyor will proceed along the second base-line from the second station to the third station in the same way as he proceeded along the first base-line from the first station to the second station; and similarly with the remaining base-lines, until he returns to the first station.

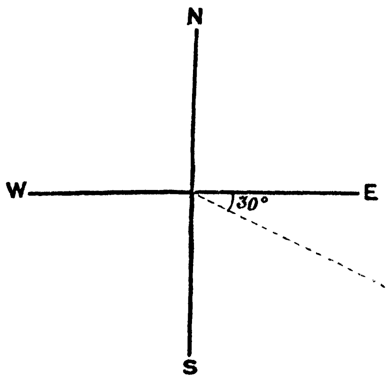
The stations are often indicated in the field-book thus: $\odot A$, $\odot B$, $\odot C$, . . . ; or thus: (1), (2), (3), . . .

The direction of a base-line may be indicated in different ways. For example—

"From $\odot$ *A* go N. to $\odot$ *B*" indicates that the base-line goes from the first station to the second station in a northerly direction.

"From (1) go S.W. to (2)" indicates that the base-line goes from the first station to the second station in a south-westerly direction.

"From (2) range E. 30° S. to (3)" indicates that the base-line goes from the second station to the third station in a direction which makes an angle of 30° with the easterly direction, reckoned towards the south (see fig.).



"From $\odot$ *B* turn left" indicates that on reaching the second station the surveyor turns to the left (not necessarily at right angles), in order to proceed from the first to the second base-line.

When the figure 0 appears in a side column, it indicates that the corresponding point on the boundary of the field is at no distance from the base-line; in other words, that at this point the base-line meets the boundary.

62. As an illustration, consider the following notes taken from a field-book.

	Links.	
	$\odot$ *B*	
0	350	
	180	75 *C*
D 60	80	
	From $\odot$ *A*	range E.

Beginning at the bottom of the central column and reading upwards, we learn—

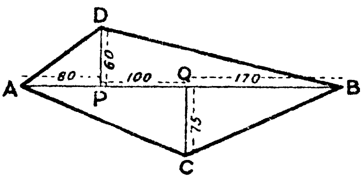
(1) That the base-line starts at *A*, and runs due east.

(2) That if we measure off 80 lks. from *A* along this base-line, and then 60 lks. to our left

in a direction at right angles to the base-line, we shall arrive at the point D, which is an angular point of the boundary.

(3) That if we measure off 180 lks. from A along the base-line, and then 75 lks. to our right in a direction at right angles to the base-line, we shall arrive at the point C, which is another angular point of the boundary.

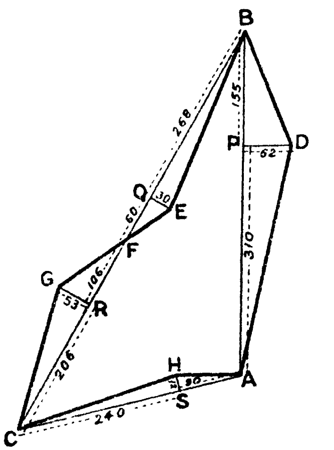
(4) That if we measure off 350 lks. from A along the base-line, we shall arrive at the point B, which is the second station.



Thus the opposite plan may be taken to represent the field indicated by these notes, where AP represents 80 lks.

,,	AQ	,,	180 ,,
,,	AB	,,	350 ,,
,,	PD	,,	60 ,,
,,	QC	,,	75 ,,

Again, consider the following notes, taken from a field-book, and the plan of the field to which they refer :—



	Links.	
	To ⊙ A	
0	330	
H 12	240	
	From ⊙ C	turn left
	To ⊙ C	
	640	0
	434	53 G
0	328	
E 30	268	
	From ⊙ B	range W. 60° S.
	To ⊙ B	
	465	0
	310	62 D
	From A	range N.

Three base-lines are here indicated, and their directions and lengths are given. Hence they can be drawn first.

The offsets from each base-line can then be drawn as in § 62.

To find the area of the figure, use the method explained in § 56.

Examination Questions—X.

The beginner is advised to work out these questions in the following order: 7, 8, 4, 12, 1, 2, 3, 6, 5, 9, 10, 11.

A. *Allahabad University: Matriculation.*

1. Make a sketch of a field from the accompanying notes, and work out its area.

	Links.	
	⊙ *A*	
0	500	
H 20	320	
G 30	140	
	0	0
	⊙ *C*	
	Turn to	the right
	⊙ *C*	
	400	0
	180	20 *F*
	0	0
	From ⊙ *B*	go N.E.
	⊙ *B*	
0	300	
E 12	200	
D 10	90	
	0	0
	From ⊙ *A*	go N.W.

2. Plan a field from the following notes, and find its area in acres, roods, and poles:—

	Links.	
	⊙ *A*	
0	409	
K 20	60	
	30	0 *H*
	20	10 *G*
	⊙ *C*	0
	Turn to	the left
	⊙ *C*	
	169	0
	30	20 *F*
	⊙ *B*	0
	Turn to	the left
	⊙ *B*	
	510	0
	160	30 *E*
	50	10 *D*
	0	0
	From ⊙ *A*	go east

3. Plan a field from the following notes, and find its area in acres, roods, and poles :—

	Links.	
	$\odot$ *A*	
	500	
	380	25 *G*
	$\odot$ *C*	
	Turn to	the right
	$\odot$ *C*	
	500	
F 175	220	
	$\odot$ *B*	
	Turn to	the right
	$\odot$ *B*	
	800	
E 100	650	
D 200	400	
	$\odot$ *A*	

B. *Punjab University: Matriculation.*

4. Sketch plan, and calculate the area of a field *ABEGFDC* from the following notes :—

	Yards.	
	To $\odot$ *G*	
	204	
To *F* 94	198	
	122	10 to *E*
To *D* 64	117	
To *C* 14	88	
	63	70 to *B*
	From $\odot$ *A*	

C. *Calcutta University: Matriculation.*

5. Draw a rough sketch of the field *ABC*, and calculate its area.

	Links.	
	$\odot$ *A*	
o	850	
8	630	
	500	0
	350	10
	$\odot$ *C*	
	Turn to	the left
	$\odot$ *C*	
0	680	
7	300	
	$\odot$ *B*	
	Turn to	the left
	$\odot$ *B*	
	510	0
	460	20
	100	15
	Begin at	$\odot$ *A* and go S.W.

D. *European Schools: Final. United Provinces.*

6 Draw a plan of the field from the following notes, and find its area:

	Links.	
	To ⊙ *D*	
	750	
	630	160 to *C*
To *E* 120	300	
	⊙ *B*	
	Turn to	the right
	⊙ *B*	
	800	
	200	250 to *C*
	From ⊙ *A*	go west

7. From the accompanying notes (measurements in links) draw a plan of the field, and find its area.

	To ⊙ *D*	
	600	
	400	120 to *C*
To *E* 100	360	
	250	140 to *B*
To *F* 160	200	
	From ⊙ *A*	

8. From the following notes in a field-book, draw a plan of the field, and find its area, the measurements being in links :—

50	To ⊙ *B*	
	300	0
	240	50
	160	0
60	100	
	80	50
0	0	40
	From ⊙ *A*	

9. Find the plan and area of a field from the accompanying field notes.

	Links.	
	⊙ *E*	
	450	
D 60	300	
	200	0 *C*
	160	40 *B*
	⊙ *A*	0
	Turn to	the left
	⊙ *A*	
	481	0
	415	30 *Q*
	360	40 *P*
N 0	320	
M 60	240	
L 30	180	
K 50	150	
0	⊙ *H*	
	Turn to	the left
	⊙ *H*	
	589	0
	450	86 *G*
	120	70 *F*
	⊙ *E*	0

10. Make a rough sketch of the field *ABC*, and calculate its area from the accompanying field-book; the chain lines are all within the field.

	Links.
10	250 ⊙ *A*
50	200
0	0
	⊙ *C*
0	390 ⊙ *C*
40	200
30	100
10	0
	⊙ *B*
0	560 ⊙ *B*
30	100
0	0
	N. 52° W.
	⊙ *A*

11. From the following notes draw a plan of the field, and find its area :—

	Links.	
	⊙ *A*	
0	818	
40	120	
0	60	20
	40	0
	⊙ *C*	0
	Turn to	the left
	⊙ *C*	
	338	0
	60	40
	⊙ *B*	0
	Turn to	the left
	⊙ *B*	
	1020	0
	320	60
	100	20
	0	0
	From ⊙ *A*	go east

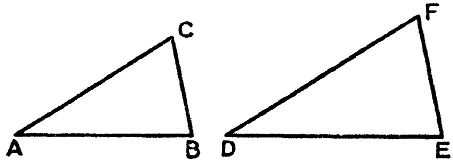
12. Draw a plan, and find the area of a field from the following notes :—

	Links.	
	To ⊙ *G*	
	1020	
To *F* 470	990	
	610	50 to *E*
To *D* 320	585	
To *C* 70	440	
	315	350 to *B*
	From ⊙ *A*	

CHAPTER XI.

ON SIMILAR FIGURES: THEIR LENGTHS.

63. FIGURES are said to be *similar* when they are of the same shape, though they need not be of the same size. Thus the $\triangle ABC$ is similar to the $\triangle DEF$.



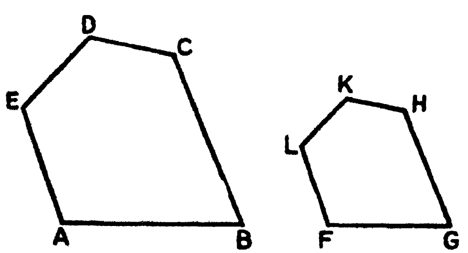
All squares are similar to one another, and so are all circles.

The plan of a field is similar to the field itself. Any object is similar to itself when magnified.

If a smaller triangle be cut off from a larger triangle by a straight line parallel to a side, the smaller triangle is similar to the larger triangle.

64. Similar rectilineal figures are equiangular, and their corresponding sides are proportionals. Thus, in the similar rectilineal figures $ABCDE$ and $FGHKL$—

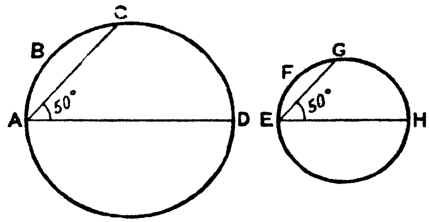
$$AB : CD = FG : HK$$



65. If any two lines, straight or curved, be drawn in a figure, and if the two corresponding lines be drawn in a similar figure,

rectilineal or curvilinear, these four lines will be proportionals. Thus in the two circles $ABCD$ and $EFGH$—

$$\text{Arc } ABC : \text{arc } EFG = AD : EH$$



Also—

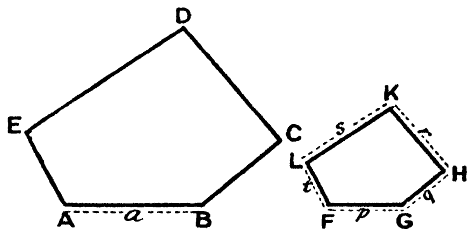
Circumference $ADCB$: diameter AD = circumference $EHGF$: diameter EH

PROPOSITION XIII.

66. *Having given one of the sides of a rectilineal figure, find the remaining sides, when all the sides of a similar rectilineal figure are given.*

Let $ABCDE$ and $FGHKL$ be two similar rectilineal figures.

Let the side AB of the figure $ABCDE$ measure a of any



linear unit. Let the corresponding side FG and the other sides GH, HK, KL, LF of the figure $FGHKL$ measure p, q, r, s, t of any linear unit respectively.

It is required to find the remaining sides of the figure $ABCDE$ in terms of a, p, q, r, s. t.

Since BC corresponds to GH, and AB corresponds to FG—

$$\therefore BC : AB = GH : FG \quad . \quad . \quad . \quad . \quad \S\ 64.$$

that is—

$$BC : a = q : p$$

Similarly, we see that—

$$CD : a = r : p$$
$$DE : a = s : p$$
$$EA : a = t : p$$

Hence rule—

Any side of a rectilineal figure is found by taking its ratio to a known side of the figure, and equating it to the ratio of the corresponding sides of a similar figure.

Or briefly—

Any side of first figure : known side of first figure = ratio of corresponding sides of second figure.

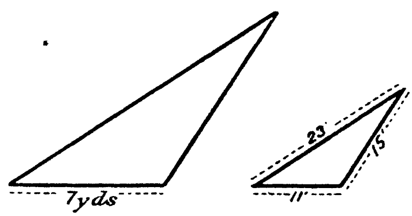
Any side of first figure : a = q : p

where p corresponds to a.

Note.—The same linear unit must be used in expressing the measures of all the sides of the same figure.

ILLUSTRATIVE EXAMPLES.

67. *Example* 1.—The sides of a triangle measure 11, 15, and 23 ft. respectively. The side of 11 ft. corresponds to a side of 7 yds. in a similar triangle: find the remaining sides of this triangle.



$\left.\begin{array}{l}\text{One remaining}\\ \text{side} : a \text{ yds.}\end{array}\right\} = q : p$ § 66.

where $a = 7$,
$q = 15$,
$p = 11$;

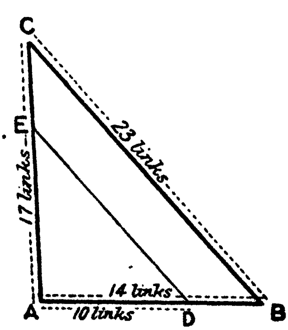
$\therefore$ one remaining side : 7 yds. $= 15 : 11$

$\therefore$ one remaining side $= 7 \text{ yds.} \times \frac{15}{11}$
$= \frac{105}{11}$ yds.
$= 9\frac{6}{11}$ yds.

The other remaining side : 7 yds. $= 23 : 11$

$\therefore$ the other remaining side $= 7 \text{ yds.} \times \frac{23}{11}$
$= \frac{161}{11}$ yds.
$= 14\frac{7}{11}$ yds.

Example 2.—In a $\triangle\, ABC$, $AB = 14$ lks., $BC = 23$ lks., $CA = 17$ lks. From a point D in the side AB a straight line is drawn parallel to BC so as to meet AC in E. If $AD = 10$ lks., find AE.



The triangles ADE and ABC are similar . . . . § 63.

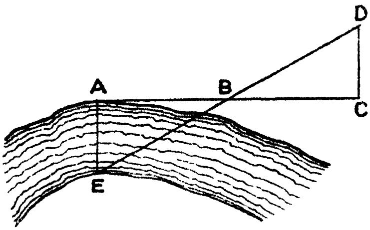
$\therefore AE : AD = AC : AB$ . . § 64.

But $AD = 10$ lks.
$AC = 17$ lks.
$AB = 14$ lks.

$\therefore AE : 10 \text{ lks.} = 17 \text{ lks.} : 14 \text{ lks.}$

$\therefore AE = 10 \text{ lks.} \times \frac{17}{14}$
$= \frac{170}{14}$ lks.
$= 12\frac{1}{7}$ lks.

Example 3—To find the breadth of an impassable river.



At a point A on the near bank of the river, which is directly opposite an object E on the far bank, draw a straight line AB at right angles to AE.

Produce AB to C, so that $BC = AB$.

At C draw CD at right angles to AC.

Produce EB to meet CD in D.

Then $\because$ triangles ABE, BCD are equal in all respects Euc. I. 26.

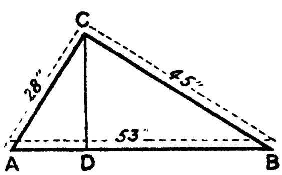
$$\therefore CD = AE$$

But CD can be measured without crossing the river.
Thus AE can be found without crossing the river.

Example 4.—The sides of a right-angled triangle measure 28, 45, and 53 in. respectively: find the length of the perpendicular from the right angle upon the hypotenuse.

Let ABC be the triangle, and let CD be the perpendicular from the right angle upon the hypotenuse.

Now, the triangles ABC, BCD, and ACD are all similar to one another; . . . . Euc. VI. 8.



$\therefore CD : CA = BC : AB$ . . § 64.

But $CA = 28$ in.

$BC = 45$ in.

$AB = 53$ in.

$\therefore CD : 28 \text{ in.} = 45 \text{ in.} : 53 \text{ in.}$

$$\therefore CD = \frac{28 \times 45}{53} \text{ in.}$$

$$= 23\tfrac{41}{53} \text{ in.}$$

Examples—XI. A.

1. The base and height of a triangle are 13 in. and 15 in. respectively: find the height of a similar triangle whose base is 9 in.

2. The base and height of a triangle are 2 ft. 3 in. and 3 ft. 9 in. respectively: find the height of a similar triangle whose base is 1 yd.

3. If a man of 5 ft. 10 in. cast a shadow of 3 ft. 2 in., what length of shadow will a man of 5 ft. 6 in. cast at the same time and place?

4. If the sides of a right-angled triangle measure 35, 37, and 12 ft. respectively, find the length of the perpendicular from the right angle upon the hypotenuse.

5. If the sides of a right-angled triangle measure 9 yds. 2 ft. 2 in., 3 yds. 1 ft., and 10 yds. 10 in. respectively, find the length of the perpendicular from the right angle upon the hypotenuse.

6. A plan of a field is 1 ft. 8 in. broad: find the width of the field if the plan is drawn to a scale of 4 ft. to the mile.

7. Two towns on a map appear 1·57 in. apart: find their actual distance from one another if the map is drawn to a scale of 170 miles to the inch.

8. The length of a country appears to be 3·7 in. on a map drawn to a scale

of 250 mi. to the inch: what length would it appear on another map drawn to a scale of 285 mi. to the inch?

9. The length of a lake is 4 mi., and on a map it appears as 0·75 in.: find its width if it appears on the same map as 0·6 in.

10. Find the height of a church spire which throws a shadow of $133\frac{1}{3}$ ft., if an upright stick 4 ft. 6 in. throws a shadow of 3 ft. 4 in. at the same time and place.

11. From a point D, in the side AB of a triangle ABC, a straight line is drawn parallel to the base BC so as to meet the side AC in E: find the length of DE if $AB = 14$ lks., $BC = 10$ lks., and $AD = 9$ lks.

12. A man, with his eye to the ground, places himself so that he can see the top of an upright stick in a line with the top of a tower: find the height of the tower if the stick measures 5 ft., and if it is placed at a distance of 9 ft. from the man, and 135 yds. from the tower.

Examples—XI. B.

13. The base and height of a triangle are 27 and 19 rasi respectively: find the base of a similar triangle whose height is 15 lathas.

14. The base and height of a triangle are 3 rasi 12 lathas, 5 rasi 9 lathas respectively: find the height of a similar triangle whose base is 7 rasi.

15. If the sides of a right-angled triangle are 48, 55, and 73 lathas, find the length of the perpendicular from the right angle upon the hypotenuse.

16. The perimeter and height of a triangle are 5 rasi, and 1 rasi 12 lathas respectively: find the perimeter of a similar triangle whose height measures 2 rasi 6 lathas.

Examination Questions—XI.

A. *Allahabad University: Matriculation.*

1. Give a practical method, by means of geometry, for ascertaining the distance of an inaccessible object A from a given position B. Illustrate your meaning by a diagram.

B. *Punjab University: Matriculation.*

2. The area of a trapezoid is 30 sq. ft., and the two parallel sides are 12 and 8 ft. respectively: find the perpendicular distance of the longer of the two parallels from the point where the two unparallels meet when produced.

C. *Punjab University: Middle School.*

3. The hypotenuse of a right-angled triangle is 123 ft., and one side is 9 yds.: find the length of the perpendicular from the right angle on the hypotenuse.

D. *European Schools: Final. United Provinces.*

4. AB and CD are two poles fixed vertically in the ground. $AB = 100$ ft., and $CD = 50$ ft. A is joined to D, and B to C, by strings, which cross at E: find the height of E above BD.

5. The radius of a circle is 20 in. From a point at a distance of 25 in. from the centre two tangents are drawn to the circle: find the distance of the point from the chord joining the points of contact.

6. The base of a triangle $= a$, and the height $= h$: find the side of a square inscribed in the triangle.

E. Roorkee Engineer: Entrance.

7. A person, wishing to find the height of a church steeple, observes that a lamp-post, 14 ft. high, is 48 ft. 9 in. distant from its base, and that from a point on the ground 11 ft. 3 in. beyond the lamp-post, in the line joining their bases, the summits of the steeple and lamp-post are in the same straight line: what is the height of the steeple?

8. The section of a canal is 32 ft. wide at the top, 14 ft. wide at the bottom, and 8 ft. deep. If the surface of the water be 26 ft. wide, what is its depth?

9. Describe method of finding approximately—

(1) The breadth of a river.

(2) The distance between two points, one of which is inaccessible and remote.

F. Roorkee Upper Subordinate: Entrance.

10. The parallel sides of a trapezoid are respectively 16 and 20 ft., and the perpendicular distance between them is 5 ft.; the other two sides are produced to meet: find the perpendicular distance of the point of intersection from the longer of the two parallel sides.

11. The parallel sides of a trapezoid are respectively 8 ft. and 14 ft.; two straight lines are drawn across the figure parallel to them, so that the four are equidistant: find the lengths of the straight lines.

G. Roorkee Engineer: Final.

12. The two legs of a right-angled triangle are 12 and 16; required the sides of the inscribed rectangle whose area is equal to half the area of the triangle.

13. The hypotenuse of a right-angled triangle is divided into two segments, measuring 30 and 20 ft. respectively, by a perpendicular from the opposite angle: find the sides of the triangle, also the area in square feet.

14. In any triangle one side a (opposite the greater angle if the triangle be obtuse or right-angled) and the perpendicular b to it from the opposite angle, are given. Find, in terms of a and b, the area of the square of which one side lies on a, and the opposite angular points on the other sides of the triangle.

H. Roorkee Upper Subordinate: Monthly.

15. The side slopes of a ditch are 5 to 2 and 7 to 2 respectively, and the breadth at the top 22 ft.: find the area of a section, supposing the sides to meet at the bottom.

I. *Additional Examination Questions.—XI.* (For Answers, see p. 168.)

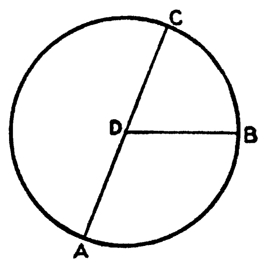
16. The breadth of the sheet of water in a tank with sloping sides is 80 cubits when the water is 6 cubits deep, and 85 cubits when the water is 10 cubits deep: what is the breadth when the water is 12 cubits deep? (Punjab University: Matriculation.)

17. A pin half an inch long at a certain distance, and a rod $7\frac{1}{2}$ ft. long at 100 yds. distance, subtend the same angle: at what distance is the pin from the point of observation? (Roorkee Upper Subordinate: Entrance.)

CHAPTER XII.

ON CIRCLES: THEIR CIRCUMFERENCES AND AREAS.

68. A *circle* is a plane figure bounded by one line which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.



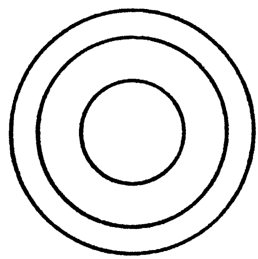
This point is called the *centre* of the circle.

A *radius* of a circle is a straight line drawn from the centre to the circumference.

A *diameter* of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

Thus in the circle ABC, BD is a radius, and AC a diameter.

Concentric circles are such as have the same centre (see fig.).



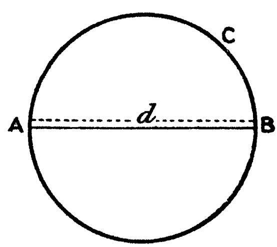
PROPOSITION XIV.

69. *To find the circumference of a circle, having given its diameter.*

Let ABC be a circle.

Let its diameter AB measure d of any linear unit.

It is required to find the circumference of the circle ABC in terms of d.



Since all circles are similar figures . § 63.

∴ the circumference of a circle bears a constant ratio to its diameter . § 65.

The value of this ratio is *incommensurable*; that is, it cannot be exactly expressed in figures, but it can be expressed with any degree of accuracy that may be required.

This ratio is denoted by the Greek letter π (*pi*).
The value of π, correct to five places of decimals, is 3·14159.
For practical purposes π is often taken equal to $\frac{22}{7}$.
Thus in all circles—

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

$$\therefore \frac{\text{circumference of circle } ABC}{AB} = \pi$$

$$\therefore \text{circumference of circle } ABC = \pi \times AB$$
$$= \pi \times d \text{ linear units}$$

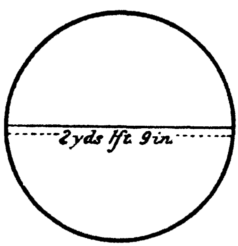
Hence rule—

The number of any linear unit in the diameter of a circle multiplied by π gives the number of the same linear unit in the circumference.

Or briefly—

$$\textbf{Circumference of a circle} = \pi \times \textbf{diameter}$$
$$\mathbf{C} = \pi \times \mathbf{d} \quad . \quad . \quad . \quad \text{(i.)}$$
$$\text{hence } \mathbf{d} = \frac{\mathbf{C}}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

ILLUSTRATIVE EXAMPLES.

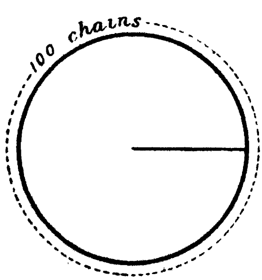


70. *Example* 1.—Find the circumference of a circle whose diameter measures 2 yds. 1 ft. 9 in. ($\pi = \frac{22}{7}$.)

Circumference of circle $= \pi d$ in. . § 69.

where $d = 7 \times 12 + 9 = 93$
and $\pi = \frac{22}{7}$;

$$\therefore \left.\begin{array}{l}\text{circumference}\\ \text{of circle}\end{array}\right\} = \frac{22}{7} \times 93 \text{ in.}$$
$$= 292\tfrac{2}{7} \text{ in.}$$
$$= 8 \text{ yds. } 0 \text{ ft. } 4\tfrac{2}{7} \text{ in.}$$



Example 2.—Find the radius of a circle whose circumference measures 100 ch. ($\pi = \frac{22}{7}$.)

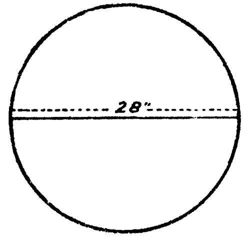
Diameter of circle $= \frac{C}{\pi}$ ch. . § 69.

where $C = 100$,
and $\pi = \frac{22}{7}$;

$$\therefore \text{diameter of circle} = \tfrac{7}{22} \times 100 \text{ ch.}$$
$$\text{and radius of circle} = \tfrac{1}{2} \text{ diameter}$$
$$\therefore \text{radius of circle} = \tfrac{7}{22} \times 50 \text{ ch.}$$
$$= 15\tfrac{10}{11} \text{ ch.}$$

Example 3.—How many revolutions will a wheel make in travelling half a mile if its diameter measure 28 in.? ($\pi = \frac{22}{7}$.)

Circumference of wheel $= \pi d$ in. . § 69.



where $d = 28$,

and $\pi = \frac{22}{7}$;

$\therefore$ circumference of wheel $= \frac{22}{7} \times 28$ in.
$= 88$ in.

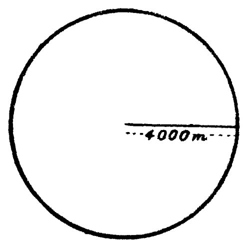
$\therefore$ $\left.\begin{array}{l}\text{number of revolutions}\\ \text{in travelling half a mile}\end{array}\right\} = \dfrac{\text{half a mile}}{88 \text{ in.}}$

$= \dfrac{880 \times 3 \times 12 \text{ in.}}{88 \text{ in.}}$

$= 360$

Example 4.—Assuming that the radius of the earth is 4000 mi., find how long it would take a man to travel round the equator at an average rate of 10 mi. an hour. ($\pi = \frac{22}{7}$.)

Length of equator $= \pi d$ mi. . . § 69.



where $d = 2 \times 4000$,

and $\pi = \frac{22}{7}$;

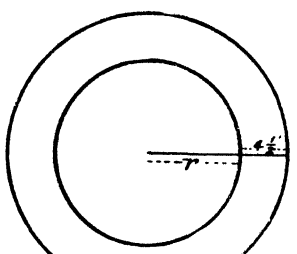
$\therefore$ $\left.\begin{array}{l}\text{length of}\\ \text{equator}\end{array}\right\} = \frac{22}{7} \times 8000$ mi.

$\therefore$ required time $= \dfrac{22 \times 8000}{7 \times 10}$ hrs.

$= \dfrac{22 \times 8000}{7 \times 10 \times 24}$ days

$= 104$ days 18 hrs. $17\frac{1}{7}$ min.

Example 5.—In turning a two-wheeled carriage once round a ring, it was observed that the outer wheel made $1\frac{3}{4}$ revolutions for every single revolution of the inner one, the wheels being 4 ft. 6 in. apart: required the circumference of the circle described by the outer wheel, the diameter of each wheel measuring 3 ft.



$\left.\begin{array}{l}\text{Circumference of outer circle}\\ \text{: circumference of inner circle}\end{array}\right\} = 1\frac{3}{4} : 1$

$= 7 : 4$

Now, if the radius of the inner circle measure r ft., the radius of the outer circle will measure $(r + 4\frac{1}{2})$ ft.

$\therefore\ 2\pi(r + 4\frac{1}{2}) : 2\pi r = 7 : 4$ . . . . § 69.

or $4(r + 4\frac{1}{2}) = 7r$

or $r = 6$

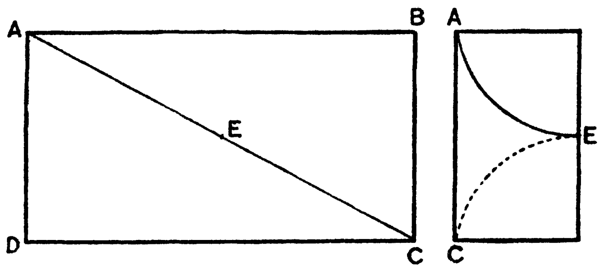
$\therefore$ radius of outer circle $= (6 + 4\frac{1}{2})$ ft.

$= 10\frac{1}{2}$ ft.

Hence—

circumference of circle described by outer wheel $= 2 \times \pi \times 10\frac{1}{2}$ ft.
$= 2 \times \frac{22}{7} \times \frac{21}{2}$ ft.
$= 66$ ft.

Example 6.—The well of a winding staircase is 6 ft. in diameter, its height to the top landing is 35 ft., and the handrail makes $3\frac{1}{2}$ revolutions: find the length. ($\pi = \frac{22}{7}$.)



Consider a rectangle $ABCD$ such that AB measures $\pi \times 6$ ft. and BC measures $(35 \div 3\frac{1}{2})$ ft. = 10 ft.

If this be bent so as to form a hollow cylinder, the diagonal AC will be seen to make a complete revolution round the cylinder.

Hence, if this cylinder be taken to represent a section of the staircase, the line AC will correspond in length and position to the handrail.

Now, according to the question, the handrail makes $3\frac{1}{2}$ revolutions.

$\therefore$ the total length of the handrail will equal length of $AC \times 3\frac{1}{2}$

$= \sqrt{AB^2 + BC^2} \times 3\frac{1}{2}$ . . . . § 16.

$= \sqrt{36\pi^2 + 100} \times 3\frac{1}{2}$ ft.

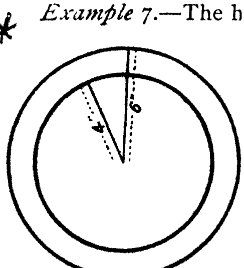
$= \sqrt{\frac{36 \times (22)^2}{7^2} + 100} \times 3\frac{1}{2}$ ft.

$= \frac{1}{2}\sqrt{22324}$ ft.

$= \frac{1}{2} \times 149\cdot412 \ldots$ ft.

$= 74\cdot706 \ldots$ ft.

Example 7.—The hands of a clock are 6 in. and 4 in. respectively: find the difference between the distances traversed by their extremities from 11 a.m. on March 3rd to 9 a.m. on May 5th. ($\pi = \frac{22}{7}$.)



Interval from March 3rd (11 a.m.) to May 5th (9 a.m.) } = 62 days 22 hrs.
= 1510 hrs.

Now, in 12 hrs. the hour-hand makes one complete revolution; therefore during the given interval, hour-hand makes $\frac{1510}{12}$ complete revolutions, and minute-hand makes 1510 complete revolutions.

$\therefore$ total distance traversed by the extremity of the hour-hand $= 2\pi \times 4 \times \frac{1510}{12}$ in.

and by the extremity of the minute-hand $= 2\pi \times 6 \times 1510$ in.

$$\therefore \text{ required difference } = 2\pi \times 1510(6 - \tfrac{1}{3}) \text{ in.}$$
$$= \frac{2 \times 22 \times 1510 \times 17}{7 \times 3} \text{ in.}$$
$$= 53784{\cdot}76 \ldots \text{ in.}$$

Example 8.—It takes a man half a minute less to cross a circular field by following the diameter than by following the circumference, walking 4 mi. an hour: find the circumference of the field. ($\pi = \frac{22}{7}$.)

Let the circumference measure x yds.

Then the diameter will measure $\frac{x}{\pi}$ yds. . . . § 69.

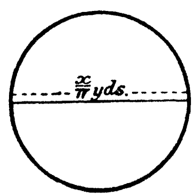
$$= \frac{7x}{22} \text{ yds.}$$

Now, the man walks 4×1760 yds. in 60 min.

$$\therefore \quad " \quad " \quad " \quad \frac{x}{2} \quad " \quad \frac{60 \times x}{4 \times 1760 \times 2} \text{ min.}$$
$$\therefore \quad " \quad " \quad " \quad \frac{7x}{22} \quad " \quad \frac{60 \times 7x}{4 \times 1760 \times 22} \text{ min.}$$

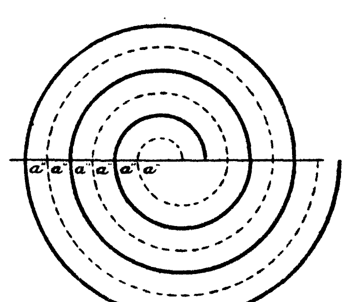
But these two times differ by half a minute.

$$\therefore \frac{60 \times x}{4 \times 1760 \times 2} = \frac{60 \times 7x}{4 \times 1760 \times 22} + \frac{1}{2}$$
$$\therefore \frac{3x}{4 \times 176} = \frac{21x}{4 \times 88 \times 22} + \frac{1}{2}$$
$$\therefore 33x = 21x + 3872$$
$$\therefore 12x = 3872$$
$$\therefore x = 322\tfrac{2}{3}$$



$\therefore$ the circumference of the field measures $322\frac{2}{3}$ yds.

Example 9.—A perfectly flexible rope of $2a$ in. in diameter is coiled closely upon the deck of a ship, and there are n complete coils: prove that the length of the rope $= \pi \,.\, an(2n + 1)$ in.



If the figure be taken to represent the coil of the rope, then the dotted line may be taken to indicate the length of the rope.

But this is seen to consist of a series of semicircles, viz.—

(1) a semicircle on $2a$ in. as diam.
(2) " " $4a$ in. "
(3) " " $6a$ in. "
(4) " " $8a$ in. "

and so on.

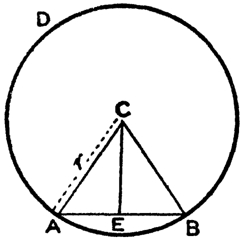
Hence—

$$\begin{aligned} \text{Length of rope} &= \pi a \text{ in.} + \pi 2a \text{ in.} + \pi 3a \text{ in} + . + \pi 2n \,.\, a \text{ in.} \quad \S\ 69. \\ &= \pi a(1 + 2 + 3 + . + 2n) \text{ in.} \\ &= \pi a \,.\, \frac{2n}{2}(2n + 1) \text{ in.} \\ &= \pi \,.\, an(2n + 1) \text{ in.} \end{aligned}$$

PROPOSITION XV.

71. *To find the area of a circle, having given its radius.*

Let ABD be a circle.



Let its radius CA measure r of any linear unit.

It is required to find the area of the circle ABD in terms of r.

Let AB be one of the sides of a regular polygon of n sides inscribed in the circle ABD.

From C, the centre of the circle, draw CE perpendicular to AB.

Join CB.

Then—

$$\begin{aligned} \text{Area of polygon} &= n \times \frac{AB \times CE}{2} \quad . \ . \ . \ . \ . \ . \ . \ . \quad \S\ 44. \\ &= \frac{n \times AB}{2} \times CE \\ &= \tfrac{1}{2}(\text{perimeter of polygon}) \times \text{radius of inscribed circle} \end{aligned}$$

But as the number of the sides of the polygon is indefinitely increased, the area of the polygon will be ultimately the area of the circle ABD, and the perimeter of the polygon will be the circumference of the circle ABD, and the radius of the inscribed circle will be the radius of the circle ABD.

Hence—

$$\text{Area of circle } ABD = \tfrac{1}{2}(\text{circum. of circle } ABD) \times (\text{radius of circle } ABD)$$

$$\text{But circumference of circle } ABD = 2\pi r \text{ linear units} \quad . \ . \quad \S\ 69.$$

$$\begin{aligned} \therefore \text{area of circle } ABD &= \tfrac{1}{2} \,.\, 2\pi r \times r \text{ sq. units} \\ &= \pi r^2 \text{ sq. units} \end{aligned}$$

Hence rule—

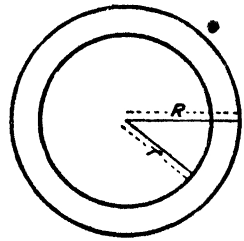
The square of the number of any linear unit in the radius of a circle multiplied by π gives the number of the corresponding square unit in the area.

Or briefly—

$$\textbf{Area of circle} = \pi(\textbf{radius})^2$$

$$\mathbf{A} = \pi \mathbf{r}^2 \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{Hence } \mathbf{r} = \sqrt{\frac{\mathbf{A}}{\pi}} \quad . \quad . \quad \text{(ii.)}$$

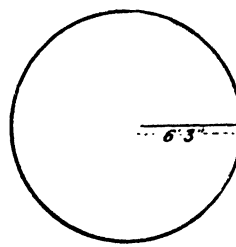


72. If R and r denote the radii of the outer and inner circles respectively which bound a *plane circular ring*, it is evident that—

$$\begin{aligned} \text{Area of ring} &= (\pi R^2 - \pi r^2) \text{ sq. units} \\ &= \pi(R^2 - r^2) \text{ sq. units} \\ &= \pi(R - r)(R + r) \text{ sq. units} \end{aligned}$$

ILLUSTRATIVE EXAMPLES.

73. *Example* 1.—Find the area of a circle whose radius measures 6 ft. 3 in. ($\pi = \frac{22}{7}$.)

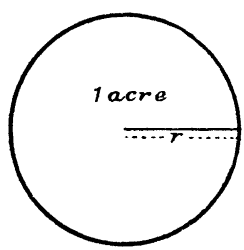


Area of circle $= \pi r^2$ sq. in. . . § 71.

where $r = 6 \times 12 + 3 = 75$,
and $\pi = \frac{22}{7}$;

$$\begin{aligned} \therefore \text{ area of circle} &= \tfrac{22}{7} \times (75)^2 \text{ sq. in.} \\ &= \tfrac{123750}{7} \text{ sq. in.} \\ &= 17678\tfrac{4}{7} \text{ sq. in.} \\ &= 122 \text{ sq. ft. } 110\tfrac{4}{7} \text{ sq. in.} \end{aligned}$$

Example 2.—Find, to the nearest inch, the radius of a circle whose area measures 1 ac. ($\pi = \frac{22}{7}$.)

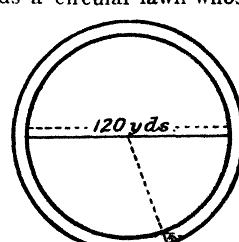


Radius of circle $= \sqrt{\frac{A}{\pi}}$ yds. . § 71.

where $A = 4840$,
and $\pi = \frac{22}{7}$;

$$\begin{aligned} \therefore \text{ radius of circle} &= \sqrt{\frac{4840 \times 7}{22}} \text{ yds.} \\ &= \sqrt{1540} \text{ yds.} \\ &= 39{\cdot}242 \ldots \text{ yds.} \\ &= 39 \text{ yds. } 0 \text{ ft. } 9 \text{ in. nearly} \end{aligned}$$

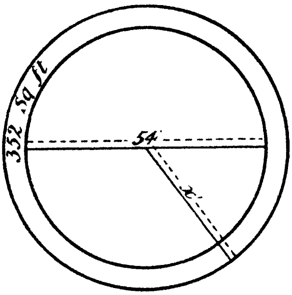
Example 3.—A path, 14 ft. wide, surrounds a circular lawn whose diameter is 120 yds.: find the area of the path. ($\pi = \frac{22}{7}$.)



Radius of inner circle $= 60 \times 3$ ft. $= 180$ ft.
radius of outer circle $= (180 + 14)$ ft. $= 194$ ft.

$$\begin{aligned} \therefore \left.\begin{array}{l}\text{area of}\\ \text{path}\end{array}\right\} &= \pi(194 - 180)(194 + 180) \text{ sq. ft.} \quad \S\, 72. \\ &= \tfrac{22}{7} \times 14 \times 374 \text{ sq. ft.} \\ &= 16{,}456 \text{ sq. ft.} \\ &= 1828 \text{ sq. yds. } 4 \text{ sq. ft.} \end{aligned}$$

Example 4.—The inner diameter of a circular building is 54 ft., and the base of the wall occupies a space of 352 sq. ft.: find the thickness of the wall. ($\pi = \frac{22}{7}$.)



Let x ft. = the outer radius of the building.

Then the space occupied by the base of the wall

$= \frac{22}{7}\{x^2-(27)^2\}$ sq. ft. . . . §72.

$\therefore \frac{22}{7}\{x^2-(27)^2\} = 352$

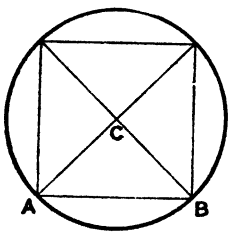
$\therefore x^2 = \frac{352 \times 7}{22} + (27)^2$

$= 841$

$\therefore x = 29$

Hence the thickness of the wall = 2 ft.

Example 5.—The area of a circle is 154 sq. in.: find the length of the side of the inscribed square. ($\pi = \frac{22}{7}$.)



Radius of circle $= \sqrt{\frac{A}{\pi}}$ in. . . § 71.

where $A = 154$,
$\pi = \frac{22}{7}$;

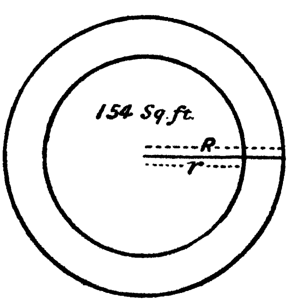
$\therefore$ radius of circle $= \sqrt{\frac{154 \times 7}{22}}$ in.

$= 7$ in.

Now, ABC is an isosceles right-angled triangle in which $AC = 7$ in.

$\therefore AB = 7\sqrt{2}$ in. . . . . § 17.
$= 7 \times 1{\cdot}414 \ldots$ in.
$= 9{\cdot}89 \ldots$ in.

Example 6.—The areas of two concentric circles are 154 sq. in. and 308 sq. in. respectively: find the breadth of the ring. (= $\frac{22}{7}$.)



Radius of larger circle $= \sqrt{\frac{A}{\pi}}$ in. § 71.

where $A = 308$,
$\pi = \frac{22}{7}$;

$\therefore$radius of larger circle $= \sqrt{\frac{308 \times 7}{22}}$ in.

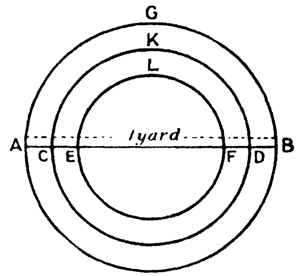
$= \sqrt{14 \times 7}$ in.
$= 7\sqrt{2}$ in.

Radius of smaller circle $= \sqrt{\frac{A}{\pi}}$ in. § 71.

where $A = 154$,
$\pi = \frac{22}{7}$;

$\therefore$ radius of smaller circle $= \sqrt{\frac{154 \times 7}{22}}$ in.
$= 7$ in.
$\therefore$ breadth of ring $= (7\sqrt{2} - 7)$ in. $= 2{\cdot}89 \ldots$ in.

Example 7.—If three men buy a grindstone one yard in diameter, how many inches of the diameter is each man entitled to grind down?



If AGB represent a section of the grindstone, then AB will be a diameter.

Area of circle $AGB = \pi r^2$ sq. in. . § 71.

where $r = \frac{36}{2} = 18$;

$\therefore$ area of circle $AGB = \pi(18)^2$ sq. in.

$\therefore$ area of portion ground down by each man $= \frac{1}{3}\pi(18)^2$ sq. in.

$\therefore$ area of circle $ELF = \frac{1}{3}\pi(18)^2$ sq. in.

i.e. $\frac{1}{4}\pi \,.\, EF^2 = \frac{1}{3}\pi(18)^2$ sq. in.

$EF = \sqrt{432}$ in.
$= 20{\cdot}784 \ldots$ in.

Again, area of inner ring $= \frac{1}{4}\pi(CD^2 - EF^2) = \frac{1}{3}\pi(18)^2$ sq. in. . § 72.

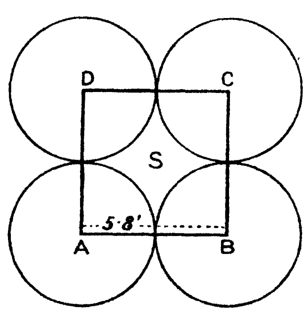
$\therefore \frac{1}{4}(CD^2 - 432) = \frac{1}{3}(18)^2$ sq. in.
$\therefore CD^2 = 864$ sq. in.
$\therefore CD = \sqrt{864}$ in.
$= 29{\cdot}393 \ldots$ in.
and $CE + FD = CD - EF$
$\therefore CE + FD = (29{\cdot}393 \ldots - 20{\cdot}784 \ldots)$ in.
$= 8{\cdot}609$ in. nearly

Again—

$AC + DB = AB - CD$
$= (36 - 29{\cdot}393 \ldots)$ in.
$= 6{\cdot}606$ in. nearly

Hence the men are entitled to grind down $6{\cdot}606 \ldots$ in., $8{\cdot}609 \ldots$ in., $20{\cdot}784 \ldots$ in. of the diameter respectively.

Example 8.—Four equal circles are described about the four corners of a square so that each touches two of the others: find the area of the space enclosed between the circumferences of the circles, each side of the square measuring 5·8 ft. ($\pi = \frac{22}{7}$.)



Since each side of the square $= 5{\cdot}8$ ft.

$\therefore$ the radius of each of the circles $= 2{\cdot}9$ ft.

$\therefore$ a quadrant of each of the circles $= \frac{1}{4} \times \pi \times (2{\cdot}9)^2$ sq. ft. § 71.

Now, the area of the enclosed space S = area of square $- 4 \times$ area of a quadrant of any one of the circles

$= (5 \cdot 8)^2$ sq. ft. $- \pi \times (2 \cdot 9)^2$ sq. ft. . . . . § 9.
$= 33 \cdot 64$ sq. ft. $- 26 \cdot 43 +$ sq. ft.
$= 7 \cdot 2$ sq. ft. nearly

Examples—XII.

Circumferences.

($\pi = \frac{22}{7}$.)

Find the circumferences of the circles having the following diameters :—

1. 21 in.
2. 18 yds. 2 ft.
3. 17 yds. 2 ft. 8 in.
4. 23 ch. 52 lks.

Find the diameters of the circles having the following circumferences :—

5. 88 in.
6. 14 yds. 2 ft.
7. 68 yds. 1 ft. 4 in.
8. 464 ch. 42 lks.

9. The wheel of a locomotive has a diameter of 30 in. : how far must it travel to make 2100 revolutions?

10. Find the cost of fencing a circular grass-plot of radius 56 ft., at the rate of 12 annas a yard.

11. What is the length of the side of a square whose perimeter is equal to the circumference of a circle 42 in. in diameter?

12. At what rate must a bicycle travel in order that its driving wheel, of 28 in. diameter, may make 540 revolutions every 5 min. ?

13. Find the thickness of a circular ring, if the inner and outer circumferences measure 4 in. and 5 in. respectively.

14. If we assume that the earth describes a circle round the sun, and that the earth is distant 95,000,000 mi. from the sun, find the distance traversed by the earth in half a year.

15. The diameter of one wheel of a bicycle is 2 in. greater than the diameter of the other wheel, and the first wheel is found to make 48 revolutions less than the other wheel in covering a distance of one mile : find the diameter of each wheel.

16. The hands of a clock are 4 in. and 3 in. respectively : find the difference between the distances traversed by their extremities in 2 days 6 hours.

17. The gauge of a circular railway is 5 ft. 6 in., and the outer wheels are each found to make 10,000 revolutions while each of the inner wheels are making 9998 revolutions : find the radius of the circle described by the inner wheels.

Areas.

($\pi = \frac{22}{7}$.)

Find the areas of the circles having the following radii :—

18. 14 yds.
19. $10\frac{1}{2}$ in.
20. 1 ft. 2 in.
21. 9 yds. 1 ft.
22. 2 yds. 1 ft. 7 in.
23 4 ch. 20 lks.
24. 2 po. 4 yds.
25. 3 ch. 7 lks.

Find the radii of the circles having the following areas :—

26. 154 sq. yds.
27. 462 sq. in.
28. 10 sq. ft. 100 sq. in.
29. 0·154 sq. ch.

30. The rent of a circular field is Rs.4620, at the rate of Rs.150 per acre: how many chains are there in the diameter?

31. A circular piece of metal, whose radius measures 1 ft. 9 in., costs Rs.77: how much is this per square inch?

32. Find the cost of turfing a circular lawn whose diameter measures 49 ft., at 12 annas per square yard.

33. Find the rent of a circular field whose diameter measures 840 yds., at Rs.15 an acre.

34. The radius of a circle is 18 ft.: find the radius of another circle whose area is one-third of the area of this circle.

35. Find the area of a circle whose circumference measures one mile.

36. Find to the nearest anna the cost of surrounding a circular grass-plot with a path of a uniform width of 4 ft., at the rate of 6 annas per square yard, if the diameter of the grass-plot is 42 ft.

37. The radii of the inner and outer circles of a ring are 24 and 25 ft. respectively: find its area.

38. The radius of the outer circle of a ring is 21 in.: find the radius of the inner circle if the area of the ring measure 26 sq. in.

39. Find the circumference of a circle whose area measures one acre.

40. Find the radius of a circle whose area is the same as that of a rectangle measuring 132 ft. by 64 ft.

41. Find the side of a square whose area is the same as that of a circle of radius 7 ft.

42. The side of a square is 42 in.: find the area of the space between the square and the inscribed circle.

43. Find the length of rope by which a horse must be tethered in order that he may be allowed to graze over 2200 sq. yards.

44. The circumference of a circle is equal to the perimeter of an equilateral triangle: compare their areas.

45. The area of a circle is equal to the area of a square: compare their perimeters.

Examination Questions—XII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Allahabad University: Matriculation.*

1. Assuming the circumference of a circle to be $3\frac{1}{7}$ times the diameter, find the circumference of the circle whose area is 1386 sq. ft.

2. A circular grass-plot 40 ft. in radius is surrounded by a ring of gravel: find the width of the gravel so that the area of the grass and gravel may be equal.

B. *Punjab University: Matriculation.*

3. Find the area of the ring, the outer and inner radii of which are 3 yds. and 5 ft. respectively, by a method other than that of subtracting the area of one circle from that of the other.

4. Find in yards correct to three places of decimals, the radius of a circle which encloses an acre.

5. Find the expense of paving a circular court 80 ft. in diameter, at 3*s*. 4*d*. per square foot, leaving in the centre a space for a fountain in the shape of a hexagon, each side of which is 1 yd. ($\pi = 3\cdot1416$.)

6. The area of a rectangular field is three-fifths of an acre, and its length is double its breadth : determine the lengths of its sides approximately

If a pony is tethered to the middle point of one of the longer sides, find the length of the tether in yards, correct to two places of decimals, in order that he may graze over half the field. ($\pi = 3\cdot1416$.)

C. *Punjab University: Middle School.*

7. The area of a circle is 385 ac. : find its circumference.

8. The radii of two circles are 6 and 8 ft. respectively : find the radius of a circle whose area is equal to the sum of the areas of the two circles.

9. Assuming that the circumference of a circle is $3\cdot1416$ times the diameter, determine, as far as four decimal places, the radius of the circle of which the circumference is 2 fur. and 60 yds.

10. The radius of the outer circle of a ring is 342 ft., and the radius of the inner circle is half of that : find the area of the ring.

11. The area of a circle is 50 sq. yds. : find the radius.

12. A road runs round a circular grass-plot ; the outer circumference is 500 yds., and the inner circumference is 300 yds. : find the area of the road.

D. *Calcutta University: Matriculation.*

13. The circumference of a circle is 100 ft. : find the length of the side of the inscribed square. The ratio of the circumference to the diameter is $3\cdot14159 : 1$. (Answer to be correct to two decimal places.)

14. A two-wheeled carriage, whose axle-tree is 4 ft. long, is driven round a circle ; the outer wheel makes one and a half revolutions for every single revolution of the inner one ; the wheels are each 3 ft. high : what is the circumference of the circle described by the outer wheel?

15. A man, by walking diametrically across a circular grass-plot, finds that it has taken him 45 sec. less than if he had kept to the path round the outside : if he walks 80 yds. a minute, what is the diameter of the grass-plot?

E. *European Schools: Final. United Provinces.*

16. The radius of a circle is $4\sqrt{2}$ ft. : find the difference in area between the square inscribed in the circle and the circle inscribed in the square.

17. A regular hexagon is inscribed in a circle of radius 1 ft. : compare the areas of the hexagon and the circle.

F. *Madras Technical: Intermediate.*

18. The difference between the circumference and diameter of a circle is 60 ft. : find the radius.

G. *Roorkee Engineer: Entrance.*

19. A circular grass-plot is surrounded by a ring of gravel b feet wide : if the radius of the circle, including the ring, be a feet, find the relation between a and b so that the areas of grass and gravel may be equal.

20. A garden in the shape of a trapezoid, whose parallel sides are 1000 and 900 yds., and the length 800 yds., has an elliptical pond in its centre, whose diameters are 300 and 400 yds. respectively : how many square poles are available for cultivation? (*Note.*—Area of ellipse $= \pi \, . \, ab$, where a and b are the semi-diameters.) ($\pi = 3\cdot1416$.)

21. A circular grass-plot, whose diameter is 40 yds., contains a gravel walk 1 yd. wide, running round it one yard from the edge : find what it will cost to turf the grass-plot at 4*d*. per square yard.

22. A wire cable is formed of six wires twisted round a central one, the diameter of each being one-eighth of an inch; the central wire is straight, and the others make one turn in 8 in.: find the length of wire required to make a yard of cable.

23. A perfectly flexible rope of $2\frac{3}{4}$ in. diameter is coiled closely upon the deck of a ship, and there are 24 complete coils: what is the length of the rope?

24. The circumferences of two concentric circles are 62·832 and 37·6992 ft.: find the area between the circles. ($\pi = 3·1416$.)

25. What will be the expense of paving a circular court of 30 ft. diameter, at 2*s*. 3*d*. per square foot, leaving in the centre a hexagonal space of $3\frac{1}{2}$ ft. side?

26. Two men, A and B, purchase a grindstone 1 yd. in diameter for Rs.15, of which the first pays Rs.8, and the other Rs.7: now, supposing the axle-hole to be 1 ft. in diameter, how many inches of the radius may A grind down before sending it to B?

27. Prove the following statement: The area of the space between two concentric circles is equal to the area of a circle which has for its diameter a chord of the outer circle which touches the inner circle.

28. In turning a chaise once round a ring, it was observed that the outer wheel made $13\frac{3}{4}$ revolutions while the inner made 11, the wheels being 4 ft. $10\frac{1}{2}$ in. asunder: required the diameter of the wheels, and of the circle made by the inner wheel.

29. The end of a room is 27 ft. wide and 18 ft. high, and has a circular window 9 ft. diameter, centre 8 ft. from floor, in it: find the length of paper, 18 in. broad, that it will take to cover it.

30. Supposing the earth were spherical, and that its circumference measured 25,000 miles, the distance from Saharunpore to Agra being about 200 mi., find how high vertically a man must ascend at one of these places in order to see the other. ($\pi = 3·1416$.)

H. *Roorkee Upper Subordinate: Entrance.*

31. A circular grass-plot, whose diameter is 70 yds., contains a gravel walk 5 yds. wide round it, 15 yds. from the edge: find what it will cost to turf the grass-plot at Rs.2 a square yard.

32. Two men, A and B, purchase a grindstone 30 in. in diameter for Rs.12, of which A pays Rs.7, and B Rs.5: now, supposing the innermost 10 in. of the diameter as useless, how many inches of the radius may A grind down before sending the grindstone to B?

33. A road runs round a circular shrubbery; the outer circumference is 500 ft., and the inner circumference is 420 ft.: find the area of the road.

34. What is the area of the largest circle that can be inscribed in a square whose area is 5,499,025 sq. ft.? Give also the length of its circumference.

35. Find the area of a gravel path, 4 ft. wide, round a circular plot whose diameter is 55 yds.

36. The well of a winding staircase is 5 ft. in diameter, its height to the top landing is 45 ft., and the handrail makes $3\frac{1}{2}$ revolutions: required its length.

37. In cutting four equal circles, the largest possible, out of a piece of cardboard 10 in. square, how many square inches must necessarily be wasted? ($\pi = 3·1416$.)

38. The inner diameter of a circular building is 68 ft. 10 in., and the thickness of the wall is 22 in.: find how many square feet of ground the base of the wall occupies. ($\pi = 3·1416$)

39. If a circle has the same perimeter as a triangle, the circle has the greater area: verify this statement in the case where the sides of a triangle are 9, 10, and 17 ft.

40. A circular pond is to be dug: what must be the length of the cord with which the circumference is to be described, so that it shall just occupy half an acre?

I. *Roorkee Engineer: Final.*

41. The radius of the inner boundary of a ring is 14 in.; the area of the ring is 100 sq. in.: find the radius of the outer boundary.

42. In driving a gig round a circular lawn, it was observed that the wheel which was kept close to the grass made only two turns, while the other wheel, which was on the gravel, made three: calculate the area of the lawn, the wheels being 5 ft. apart on the axle. Answer to be given in square feet. ($\pi = 3{\cdot}14159$.)

43. A train runs in a circle whose radius is 2 mi.: if the gauge is 5 ft. 6 in., and the circuit is made in 40 min., how many miles an hour do the outer wheels move faster than the inner ones?

44. The well of a winding staircase is 7 ft. in diameter, its height to the top landing is 36 ft., and the handrail makes three revolutions: required its length.

J. *Roorkee Upper Subordinate: Monthly.*

45. A road runs round a circular plot of ground; the outer circumference of the road is 44 yds. longer than the inner: find the breadth of the road.

46. A horse is tethered by a chain fastened to a ring which slides on a rod bent into the form of a triangle: find the area outside the triangle over which he can graze, the sides of the triangle being 30, 40, and 50 ft. respectively, and the length of the chain 15 yds.

47. If a regular hexagon be inscribed in a circle, of which the circumference is 10 ft.: find the area of the space enclosed between them.

K. *Sandhurst.*

48. Find in feet, to three places of decimals, the radius of a circle the area of which is equal to the area of a regular hexagon, the side of which is 2 ft. ($\pi = 3{\cdot}1416$.)

49. 105 halfpennies lying on a flat surface, with their edges in contact, are just contained by a frame in the form of an equilateral triangle: the diameter of a halfpenny being 1 in., show that the side of the triangle is $(13 + \sqrt{3})$ in., and calculate its area approximately.

50. Assuming that $\pi = 3{\cdot}1416$, find the radius and the perimeter of a circle, the area of which is $5{\cdot}309304$ sq. ft.

L. *Additional Examination Questions.—XII.* (For Answers, see p. 168.)

51. Two equal circles of diameter 9 in. touch one another, and from the point of contact as centre a third circle of radius 9 in. is drawn: find the radius and the area of the circle inscribed in either of the two spaces enclosed by the three given circles. (Roorkee Upper Subordinate: Entrance.)

52. The times taken by a cyclist going at a steady rate respectively round the outer and inner edges of a circular track are as 23 to 22, and the width of the track is 15 ft.: find the diameter of the circle forming the inner edge of the track. (Roorkee Engineer: Entrance.)

53. In a circle of unit radius a regular hexagon is inscribed, and in the hexagon another circle: find correct to four decimal places the area of the ring enclosed between the two circles. $\pi = 3{\cdot}14159$. (Punjab University: Matriculation.)

54. Find to four figures the ratio of the perimeters of a circle and a regular hexagon of the same area. $\pi = 3{\cdot}14159$. (Roorkee Engineer: Entrance.)

CHAPTER XIII.

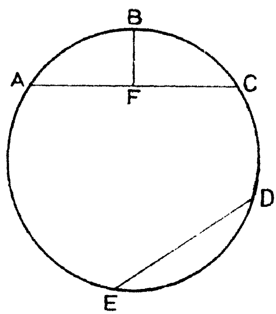
ON CIRCLES: THEIR CHORDS AND ARCS.

74. A *chord* of a circle is a straight line joining two points on the circumference.

An *arc* of a circle is a part of the circumference.

The straight line joining the extremities of an arc is called the *chord of the arc.*

The perpendicular from the middle point of an arc upon the chord of the arc is called the *height of the arc.*

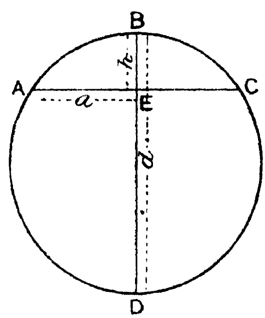


Thus in the circle $ABCDE$, ED is a chord, ABC is an arc, AC is the chord of the arc ABC, BF is the height of the arc ABC.

PROPOSITION XVI.

75. *To find the chord of an arc of a circle, having given the height of the arc and the diameter of the circle.*

Let ABC be an arc of the circle $ABCD$.



Let BE, the height of the arc, and BD, the diameter of the circle, measure h and d of the same linear unit respectively.

It is required to find the chord of the arc in terms of h and d.

The area of the rectangle contained by BE and ED = the area of the rectangle contained by AE and EC

Euc. III. 35.

But the area of the rectangle contained by BE and ED measures $h(d - h)$ of the corresponding square unit . . § 8.

$\therefore$ rectangle $AE \,.\, EC = h(d - h)$ square units

That is—

$$\text{square on } AE = h(d-h) \text{ square units (since } AE = EC)$$
$$\therefore AE = \sqrt{h(d-h)} \text{ linear units} \quad . \quad . \quad . \quad . \quad \S\ 9.$$
$$\therefore AC = 2\sqrt{h(d-h)} \quad \text{,,}$$

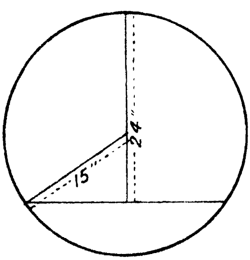
Hence rule—

Multiply the number of any linear unit in the height of an arc by the difference between this number and the number of the same linear unit in the diameter of the circle; then twice the square root of the product gives the number of the same linear unit in the chord of the arc.

Or briefly—

$$\textbf{Chord of an arc} = 2\sqrt{\textbf{height} \times (\textbf{diameter} - \textbf{height})}$$
$$\mathbf{2a = 2\sqrt{h(d-h)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$
$$\text{Hence } \mathbf{d = h + \frac{a^2}{h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.})$$
$$\text{and } \mathbf{h = \frac{d \pm \sqrt{d^2 - 4a^2}}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iii.})$$

ILLUSTRATIVE EXAMPLES.

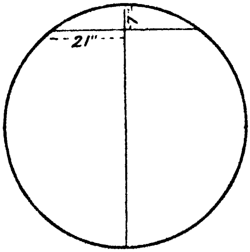


76. *Example* 1.—Find the chord of an arc whose height is 24 in., in a circle of radius 15 in.

$$\text{Chord of arc} = 2\sqrt{h(d-h)} \text{ in.} \quad . \quad \S\ 75.$$

where $h = 24$,
  and $d = 30$;

$$\therefore \text{chord of arc} = 2\sqrt{24 \times 6} \text{ in.}$$
$$= 2\sqrt{144} \text{ in.}$$
$$= 24 \text{ in.}$$



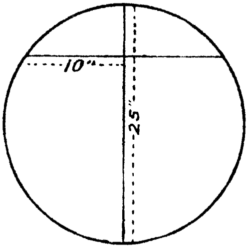
Example 2.—The height of an arc is 7 in., and its chord is 42 in.: find the diameter of the circle.

$$\text{Diameter of circle} = \left(h + \frac{a^2}{h}\right) \text{in.} \quad . \quad \S\ 75.$$

where $h = 7$,
  and $a = 21$;

$$\therefore \text{diameter of circle} = \left(7 + \frac{21 \times 21}{7}\right) \text{in.}$$
$$= 70 \text{ in.}$$

Example 3.—In a circle of diameter 25 in., the chord of an arc is 20 in.: find its height.



Height of arc $= \dfrac{d \pm \sqrt{d^2 - 4a^2}}{2}$ in. § 75.

where $d = 25$,
and $a = 10$;

$\therefore$ height of arc $= \dfrac{25 \pm \sqrt{(25-20)(25+20)}}{2}$ in.

$= \dfrac{25 \pm \sqrt{225}}{2}$ in.

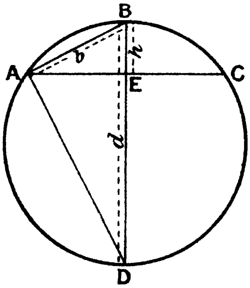
$= \dfrac{25 \pm 15}{2}$ in.

$= 20$ in. or 5 in.

These two answers obviously refer to the two arcs into which the circumference of the circle is divided by the chord.

PROPOSITION XVII.

77. *To find the chord of half an arc of a circle, having given the height of the arc and the diameter of the circle.*



Let ABC be an arc of the circle $ABCD$.

Let BE, the height of the arc, and BD, the diameter of the circle, measure h and d of the same linear unit respectively.

It is required to find the chord of half the arc in terms of h and d.

Join AB and AD.

$\because$ the triangles ABE, ABD are similar
Euc. III. 31 and VI. 8.

$\therefore DB : AB = AB : BE$ . . . . § 64.

$\therefore$ area of square on AB = area of rectangle contained by DB and BE . . . Euc. VI. 17.

But the rectangle contained by DB and BE measures dh of the corresponding square unit . . . . § 8.

$\therefore$ area of square on $AB = dh$ square units

$\therefore AB = \sqrt{dh}$ linear units . § 9.

Hence rule—

Multiply the number of any linear unit in the height of an arc by the number of the same linear unit in the diameter of the circle; then the square root of the product gives the number of the same linear unit in the chord of half the arc.

Or briefly—

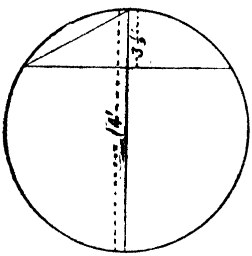
Chord of half an arc = $\sqrt{\text{diameter of circle} \times \text{height of arc}}$

$$\mathbf{b} = \sqrt{\mathbf{dh}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{Hence } \mathbf{d} = \frac{\mathbf{b}^2}{\mathbf{h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$\text{and } \mathbf{h} = \frac{\mathbf{b}^2}{\mathbf{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

ILLUSTRATIVE EXAMPLES.



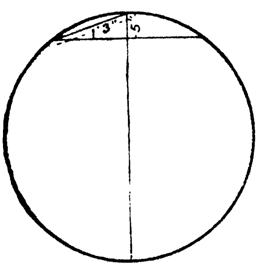
78. *Example* 1.—The height of an arc is $3\frac{1}{2}$ ft., and the diameter of the circle is 14 ft. : find the chord of half the arc.

Chord of half the arc $= \sqrt{dh}$ ft. § 77.

where $d = 14$,
and $h = 3\frac{1}{2}$;

$\therefore$ chord of half the arc $= \sqrt{14 \times \frac{7}{2}}$ ft.
$= \sqrt{49}$ ft.
$= 7$ ft.

Example 2.—The height of an arc is 5 in., and the chord of half the arc is 1 ft. 3 in. : find the diameter of the circle.

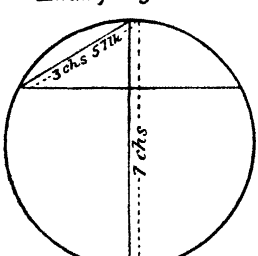


Diameter of circle $= \frac{b^2}{h}$ in. . § 77.

where $b = 15$,
and $h = 5$;

$\therefore$ diameter of circle $= \frac{15 \times 15}{5}$ in.
$= 45$ in.
$= 3$ ft. 9 in.

Example 3.—Find the height of an arc of a circle, when the chord of half the arc is 3 ch. 57 lks., and the diameter of the circle is 7 ch.

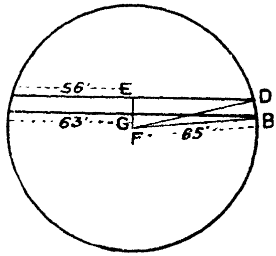


Height of arc $= \frac{b^2}{d}$ ch. . . § 77.

where $b = 3{\cdot}57$,
and $d = 7$;

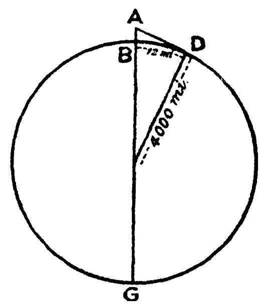
$\therefore$ height of arc $= \frac{3{\cdot}57 \times 3{\cdot}57}{7}$ ch.
$= 0{\cdot}51 \times 3{\cdot}57$ ch.
$= 1{\cdot}8207$ ch.

Example 4.—In a circle of radius 65 ft., two parallel chords are drawn on the same side of the centre, measuring 126 ft. and 112 ft. respectively: find the distance between them.



$$\begin{aligned} GE &= \text{distance between parallel chords} \\ &= FE - FG \\ &= \sqrt{FD^2 - ED^2} - \sqrt{FB^2 - GB^2} \quad \S\ 16. \\ &= [\sqrt{(65)^2 - (56)^2} - \sqrt{(65)^2 - (63)^2}] \text{ ft.} \\ &= (33 - 16) \text{ ft.} \\ &= 17 \text{ ft.} \end{aligned}$$

Example 5.—Assuming that the radius of the earth is 4000 mi., find approximately how many feet above the earth's surface a person must ascend in order to just see an object on the earth's surface distant 12 mi. from him.



A person stationed at B must ascend to A in order that he may just see an object placed at D.

$$\begin{aligned} AD^2 &= AG \times AB \quad . \text{ Euc. III. 36.} \\ &= (AB + BG) \times AB \end{aligned}$$

Now, AD may be taken equal to the arc $BD = 12$ mi., and if h denote the number of feet in AB, $\frac{h}{5280}$ will denote the number of miles.

Thus we have—

$$\begin{aligned} (12)^2 &= \left(\frac{h}{5280} + 8000\right) \times \frac{h}{5280} \\ &= \left(\frac{h}{5280}\right)^2 + \frac{8000 \times h}{5280} \end{aligned}$$

But h is small compared with 5280, therefore $\frac{h}{5280}$ is a small fraction; therefore $\left(\frac{h}{5280}\right)^2$ is a still smaller fraction, and may be neglected.

Hence—

$$\begin{aligned} (12)^2 &= \frac{8000 \times h}{5280} \text{ approximately} \\ \text{or } h &= \frac{144 \times 5280}{8000} \text{ approximately} \\ &= 95{\cdot}04 \text{ approximately} \end{aligned}$$

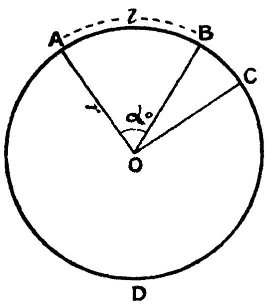
Thus—

$$\text{required height} = 95{\cdot}04 \text{ ft.}$$

Note.—It is only when h is small compared with 5280 that we can neglect the fraction $\left(\frac{h}{5280}\right)^2$.

PROPOSITION XVIII.

79. *To find the length of an arc of a circle, having given the radius of the circle and the angle subtended by the arc at the centre.*



Let AB be an arc of the circle $ABCD$.

Let OA, the radius of the circle $ABCD$, measure r of any linear unit.

Join OB.

Let AOB, the angle subtended by the arc at the centre, measure a°.

It is required to find the length of the arc in terms of r and a.

Draw OC at right angles to OA.

In any circle arcs are proportional to the angles which they subtend at the centre . . . . Euc. VI. 33.

$\therefore$ in the circle $ABCD$

$$\text{arc } AB : \text{arc } ABC = \angle AOB : \angle AOC$$

$$\text{But } \angle AOC = 90^\circ$$

$\therefore$ arc ABC is the fourth part of the circumference

$$\therefore \text{arc } AB : \frac{\text{circumference of circle}}{4} = \angle AOB : 90^\circ$$

$$\text{That is, arc } AB : \frac{2\pi r}{4} = a^\circ : 90^\circ \quad . \quad . \quad . \quad . \quad \S\ 69.$$

$$\therefore \text{arc } AB = \frac{a^\circ}{360^\circ} \times 2\pi r \text{ linear units}$$

Hence rule—

The length of an arc is obtained by multiplying the circumference of the circle by $\frac{a}{360}$, *where* a° *is the central angle of the arc.*

Or briefly—

$$\textbf{Length of arc} = \frac{\textbf{central angle of arc}}{\mathbf{360^\circ}} \times \textbf{circumference of circle}$$

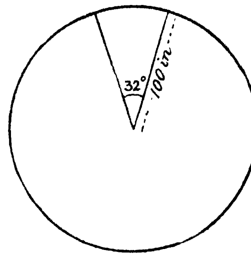
$$l = \frac{a}{360} \times 2\pi r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i.)$$

$$\text{Hence } a = 360 \times \frac{1}{2\pi r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii.)$$

$$\text{and } r = \frac{180}{\pi} \times \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii.)$$

ILLUSTRATIVE EXAMPLES.

80. *Example* 1.—The radius of a circle is 100 in.: find the length of an arc which subtends an angle of 32° at the centre. ($\pi = \frac{22}{7}$.)

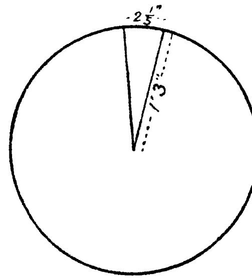


$$\text{Length of arc} = \frac{a}{360} \times 2\pi r \text{ in.} \quad \S\ 79.$$

where $r = 100$,
$a = 32$,
$\pi = \frac{22}{7}$;

$$\therefore \text{length of arc} = \frac{32}{360} \times \frac{2 \times 22 \times 100}{7} \text{ in.}$$
$$= \frac{3520}{63} \text{ in.}$$
$$= 55\tfrac{55}{63} \text{ in.}$$

Example 2.—The radius of a circle is 1 ft. 3 in.: find the angle subtended at the centre by an arc of $2\frac{1}{5}$ in. ($\pi = \frac{22}{7}$.)

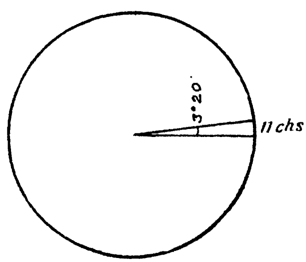


$$\text{Central angle} = 360° \times \frac{l}{2\pi r} \quad \S\ 79.$$

where $l = 2\frac{1}{5}$,
$r = 15$,
$\pi = \frac{22}{7}$;

$$\therefore \text{central angle} = 360° \times \frac{11 \times 7}{5 \times 2 \times 22 \times 15}$$
$$= \frac{42}{5} \text{ deg.}$$
$$= 8°\ 24'$$

Example 3.—The length of an arc of a circle is 11 ch., and it subtends an angle of 3° 20′ at the centre: find the radius. ($\pi = \frac{22}{7}$.)



$$\text{Radius of circle} = \frac{180}{\pi} \times \frac{l}{a} \text{ch.} \quad \S\ 79.$$

where $l = 11$,
$a = 3\frac{1}{3}$,
$\pi = \frac{22}{7}$;

$$\therefore \left.\begin{array}{l}\text{radius of}\\ \text{circle}\end{array}\right\} = \frac{180 \times 7 \times 11 \times 3}{22 \times 10} \text{ ch.}$$
$$= 189 \text{ ch.}$$

PROPOSITION XIX.

81. *To find the length of an arc of a circle, having given the chord of the arc and the chord of half the arc.*

Rule—

From eight times the chord of half the arc subtract the chord of the arc; then one-third the remainder will give the length of the arc.

Or briefly—

$$\text{Arc of circle} = \frac{8 \times \text{chord of semi-arc} - \text{chord of arc}}{3}$$

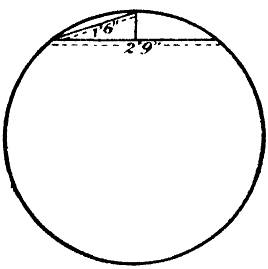
$$l = \frac{8b - 2a}{3}$$

The proof of this formula involves a knowledge of the higher mathematics, and is for this reason omitted.

This formula only gives a close approximation to the true length of the arc. The error diminishes as the central angle of the arc diminishes. When, therefore, the arc has a large central angle, it is advisable to find the length of half the arc by means of this formula, and then double the result.

ILLUSTRATIVE EXAMPLES.

82. *Example* 1.—The chord of an arc is 2 ft. 9 in., and the chord of half the arc is 1 ft. 6 in.: find the length of the arc.



Length of arc $= \frac{8b - 2a}{3}$ in. . § 81.

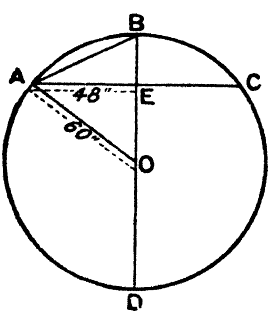
where $2a = 33$,
and $b = 18$;

$\therefore$ length of arc $= \frac{144 - 33}{3}$ in.

$= \frac{111}{3}$ in.

$= 37$ in. $= 3$ ft. 1 in.

Example 2.—Find the length of an arc whose chord measures 96 in., in a circle of radius 60 in.



Let $ABCD$ be the given circle.
Then we know that—

$OA = 60$ in.
$AE = 48$ in.
$\therefore OE = \sqrt{(60)^2 - (48)^2}$ in. . § 16.
$= \sqrt{12 \times 108}$ in.
$= 36$ in.
$\therefore BE = (60 - 36)$ in.
$= 24$ in.
$\therefore AB = \sqrt{(48)^2 + (24)^2}$ in. . § 16.
$= 53{\cdot}665 \ldots$ in.

Now, length of arc $ABC = \frac{8b - 2a}{3}$ in. § 81

where $b = 53{\cdot}665 \ldots$
and $a = 48$;

$\therefore$ length of arc $ABC = \frac{429{\cdot}32 \ldots - 96}{3}$ in.

$= 111{\cdot}10 \ldots$ in.

In working out examples on the chords of circles, it will be found advisable to draw a figure in each case and investigate it separately, rather than trust to remembering the formulæ.

Examples—XIII.

Chords of a Circle.

$(\pi = \frac{22}{7}.)$

1. The height of an arc is 8 ft., and the diameter of the circle 40 ft.: find the chord of the arc.

2. The height of an arc is 9 in., and the chord of the arc is 2 ft. 6 in.: find the diameter of the circle.

3. The chord of an arc is 7 ch. 50 lks., and the diameter of the circle is 12 ch. 50 lks.: find the height of the arc.

4. The height of an arc is 1 ft. 9 in., and the diameter of the circle is 2 yds. 1 ft.: find the chord of half the arc.

5. The height of an arc is 7 ch., and the chord of half the arc is 12 ch. 60 lks.: find the diameter of the circle.

6. The chord of an arc is 4 yds. 2 ft., and the height of the arc is 1 ft. 1 in.: find the chord of half the arc.

7. The height of an arc is 3·6 ch., and the chord of half the arc is 8·5 ch.: find the chord of the arc.

8. The chord of an arc is 4 ch. 40 lks., and the chord of half the arc is 2 ch. 21 lks.: find the height of the arc.

9. Prove that—

$$d = \frac{b^2}{\sqrt{b^2 - a^2}},$$

where a denotes half the chord of an arc, b the chord of half the arc, and d the diameter of the circle.

10. The chord of an arc is 9 in., and the chord of half the arc is $7\frac{1}{2}$ in.: find the diameter of the circle.

11. The height of an arc is 2 ft. 3 in., and the chord of half the arc is 5 ft. 3 in.: find the distance of the chord of the arc from the centre of the circle.

Arcs of a Circle.

$(\pi = \frac{22}{7}.)$

12. The radius of a circle is 1 ft. 4 in.: find the length of an arc which subtends an angle of 30° at the centre.

13. The radius of a circle is 7 ft. 7 in.: find the length of an arc which subtends an angle of 45° at the centre.

14. The radius of a circle is 2 ft. 11 in.: find the length of an arc which subtends an angle of 7° 12′ at the centre.

15. The radius of a circle is 6 in.: find the angle subtended at the centre by an arc of 11 in.

16. The radius of a circle is 45 lks.: find the angle subtended at the centre by an arc of 33 lks.

17. The radius of a circle is 10 ft.: find the angle subtended at the centre by an arc of 4 ft. 7 in.

18. The length of an arc of a circle is 77 lks., and it subtends an angle of 2° 15′ at the centre: find the radius.

19. The length of an arc of a circle is 2 yds. 2 ft. 3 in., and it subtends an angle of $7^\circ\ 30'$ at the centre : find the radius.

20. The chord of an arc is 1 ft. 7 in., and the chord of half the arc is 11 in. : find approximately the length of the arc.

21. The chord of an arc is 6 ch. 24 lks., and the chord of half the arc is 4 ch. 32 lks. : find approximately the length of the arc.

22. The chord of an arc is 48 in., and the radius of the circle is 30 in. : find the length of the arc.

23. The chord of an arc is 2 ft. 8 in., and the height of the arc is 1 ft. : find approximately the length of the arc.

Examination Questions—XIII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Allahabad University : Matriculation.*

1. The difference between the areas of two squares inscribed and circumscribed about a circle is 338 sq. ft. : find the radius of the circle.

B. *Punjab University : Middle School.*

2. The chord of an arc is 5 ft., and the diameter of the circle is 7 ft. : find the height of the arc in inches to four decimal places.

3. The chord of an arc is 8 yds., and the chord of half the arc is 13 ft. : find the diameter of the circle.

4. The chord of an arc is 10 ft., and the height of the arc is 2 yds. : find the diameter of the circle, and the chord of half the arc.

C. *Calcutta University : Matriculation.*

5. The chord of an arc is 100 ft., and the angle subtended by it on the circumference is 150° : find the radius of the circle, the height of the arc and the chord of half the arc.

D. *European Schools : Final. United Provinces.*

6. The chord of an arc is 36 ft., and the chord of half the arc is $19\frac{1}{2}$ ft. : find the diameter of the circle.

E. *Madras Technical : Elementary.*

7. Find the length of an arc of a circle whose radius is 6 ft., the chord of the arc being 8 ft.

F. *Madras Technical : Intermediate.*

8. Find the arc of a circle whose radius is 8 ft., the chord of the arc being 12 ft.

G. *Roorkee Engineer : Entrance.*

9. The tops of two vertical rods on the earth's surface, each of which is 10 ft. high, cease to be visible from each other when they are 8 mi. apart : what is the earth's radius ?

10. O is the middle point of a straight line AB, 10 in. long, and from O as centre a circle is described with radius 7 in. ; P is a point on the circumference, such that $PA = 5$ in. : find PB.

11. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100 : what is the diameter of the semicircle ?

12. Given the chord 20 ft., and height 4 ft., of an arc of a circle, find the diameter and length of arc.

H. *Roorkee Upper Subordinate: Entrance.*

13. AB and AC are the chords of a circle at right angles to one another; their lengths are 30 ft. and 40 ft. respectively: find the height of the arc AC, and the diameter of the circle.

14. The chord of half an arc is 2 ft. 6 in., and the diameter of the circle is 4 ft. 2 in.: find the chord of the arc.

15. The radius of a circle is 7 ft.: find the perpendicular from the centre on a chord 8 ft. long.

16. A segment of a circular ring is $2\frac{1}{2}$ ft. thick, the chord of the inner arc is equal to the radius: find the mean length of the segment. The length of the radius is 10 ft.

17. The chord of an arc is 49 ft., and the chord of half the arc is 25 ft.: find the diameter of the circle.

18. Find the length of the minute hand of a clock, the extremity of which moves over an arc 5 in. in length in $3\frac{1}{4}$ min.

19. The height of an arc was found by measurement to be 7 ft. $9\frac{1}{2}$ in., and the chord of half the arc 15 ft. 7 in.: with what radius had the arc been described?

I. *Roorkee Engineer: Final.*

20. Two parallel chords in a circle are 6 in. and 8 in. long, and 1 in. apart: find the diameter.

21. The chord of an arc is 24 ft., and the height 5 ft.: find the length of the arc.

22. The diameter of a circle is 12 ft.: find the side of the square inscribed in it.

23. Find the area of a square inscribed in a quadrant whose radius is $\sqrt{3}$, two sides of the square being coincident with the radii.

J. *Roorkee Upper Subordinate: Monthly.*

24. The span of a bridge, the form of which is an arc of a circle, being 96 ft., and the height 12 ft., find the radius.

25. The width of a circular walk is 4 ft., and the length of the line which is a chord of the outer circumference and a tangent to the inner circumference, is 20 ft: find the area of the walk. ($\pi = 3{\cdot}14159$.)

K. *Additional Examination Questions.—XIII.* (For Answers, see p. 168.)

26. Calculate the distance of the horizon from the eye of an observer standing on the Kutab Minar, the height of eye from the ground being 240 ft., and the size of the earth 8000 miles through. (Punjab University: Matriculation.)

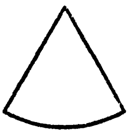
27. Four circles are each 1 in. in diameter. They are placed so that two of them touch two of the others, and the remaining two touch three of the others: find the area of the rhombus whose angles are at their centres. (Superior Accounts: 4th Grade.)

28. AB is a diameter of a circle; BC is a chord subtending an angle of 30° at A; CD is drawn perpendicular to AB. If $AC = 10\sqrt{3}$ in., find AB and AD. (Roorkee Upper Subordinate: Entrance.)

29. A lighthouse is to be constructed at a distance of 42 miles from a port: how high should it be in order that the light may be just visible from the port, taking the average height of a man as 6 ft.? (Roorkee Upper Subordinate: Monthly.)

CHAPTER XIV.

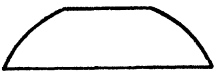
ON CIRCLES: THEIR SECTORS AND SEGMENTS.



83. A *sector* of a circle is a figure bounded by two radii and the arc intercepted between them (see fig.).

The angle contained by the two radii is the *angle of the sector.*

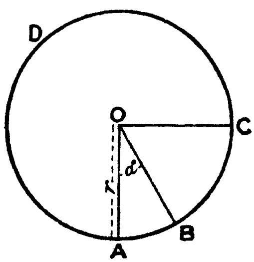
A *segment* of a circle is a figure bounded by a chord and the part of the circumference it cuts off (see fig.).



A *zone* of a circle is a figure bounded by two parallel chords and their intercepted arcs (see fig.).

PROPOSITION XX.

84. *To find the area of a sector of a circle, having given the radius of the circle and the angle of the sector.*



Let OAB be a sector of the circle $ABCD$.

Let OA, the radius of the circle $ABCD$, measure r of any linear unit. Let AOB, the angle of the sector, measure a°.

It is required to find the area of the sector in terms of r and a.

Draw OC at right angles to OA.

In any circle the areas of sectors are proportional to their angles . . . . Euc. VI. 33.

$\therefore$ in the circle $ABCD$, area of sector OAB : area of sector OAC $\Big\} = \angle AOB : \angle AOC$

But $\angle AOC = 90^\circ$

$\therefore$ sector AOC is the fourth part of the circle $ABCD$

$$\therefore \text{ area of sector } OAB : \frac{\text{area of circle } ABCD}{4} = \angle AOB : 90^\circ$$

That is, area of sector $OAB : \dfrac{\pi r^2}{4}$ sq. units $= a^\circ : 90^\circ$ § 71.

$$\therefore \text{ area of sector } OAB = \frac{a^\circ}{360^\circ} \times \pi r^2 \text{ sq. units}$$

Hence rule—

The area of a sector is obtained by multiplying the area of the circle by $\frac{a}{360}$, where a° is the angle of the sector.

Or briefly—

$$\textbf{Area of sector} = \frac{\textbf{angle of sector}}{\mathbf{360^\circ}} \times \textbf{area of circle}$$

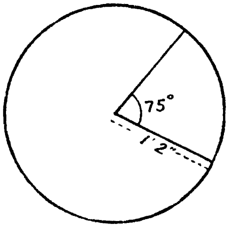
$$\mathbf{A} = \frac{a}{360} \cdot \pi \mathbf{r}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{Hence } a = 360 \cdot \frac{\mathbf{A}}{\pi \mathbf{r}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$\text{and } \mathbf{r} = \sqrt{\frac{360}{a} \cdot \frac{\mathbf{A}}{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

ILLUSTRATIVE EXAMPLES.

85. *Example* 1.—In a circle of radius 1 ft. 2 in. find the area of a sector whose angle measures 75°. ($\pi = \frac{22}{7}$.)

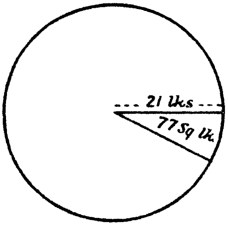


Area of sector $= \frac{a}{360} \pi r^2$ sq. in. . § 84.

where $a = 75$,
$\pi = \frac{22}{7}$,
$r = 14$;

$\therefore$ area of sector $= \frac{75}{360} \times \frac{22}{7} \times (14)^2$ sq. in.
$= 128\frac{1}{3}$ sq. in.

Example 2.—The area of a sector is 77 sq. lks.; the radius is 21 lks.: find the angle of the sector. ($\pi = \frac{22}{7}$.)

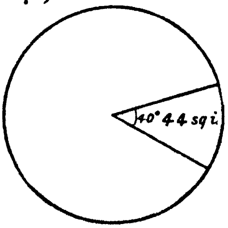


The angle of the sector $= 360^\circ \times \frac{A}{\pi r^2}$ § 84.

where $A = 77$,
$\pi = \frac{22}{7}$,
$r = 21$;

$\therefore$ angle of sector $= 360^\circ \times \frac{77 \times 7}{22 \times 21 \times 21}$
$= 20^\circ$

Example 3.—The area of a sector is 44 sq. in.; the angle of the sector is 40°: find the radius of the sector. ($\pi = \frac{22}{7}$.)

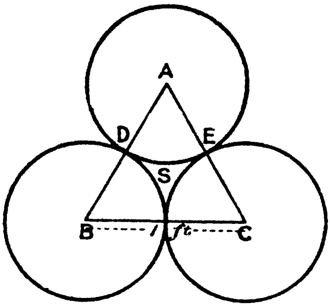


Radius of sector $= \sqrt{\frac{360}{a} \times \frac{A}{\pi}}$ in. § 84.

where $a = 40$,
$A = 44$,
$\pi = \frac{22}{7}$;

$\therefore$ radius of sector $= \sqrt{\frac{360}{40} \times \frac{44 \times 7}{22}}$ in.
$= \sqrt{126}$ in.
$= 11{\cdot}225$ in. nearly

Example 4.—Three equal circles are so placed that the circumference of each touches the other two. If the diameter of each circle measure one foot, find the area of the unoccupied space between them. ($\pi = \frac{22}{7}$.)



If the centres A, B, C of the circles be joined as in the figure, it is evident that ABC is an equilateral triangle of side = 1 ft.

$\therefore$ area of equilateral $\triangle ABC$ $= \frac{\sqrt{3}}{4}$ sq. ft. . § 21.

Now, unoccupied space S = area of $\triangle ABC$ $- 3 \times$ area of sector ADE

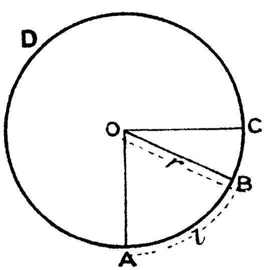
and area of sector $ADE = \frac{1}{6} \times \pi \times (\frac{1}{2})^2$ sq. ft. . . . § 71.

(since $\angle BAC = 60^\circ$)

$$\therefore \text{the unoccupied space } S = \frac{\sqrt{3}}{4} \text{ sq. ft.} - \tfrac{1}{2} \times \tfrac{22}{7} \times \tfrac{1}{4} \text{ sq. ft.}$$
$$= 36\sqrt{3} \text{ sq. in.} - 56{\cdot}571428 \text{ sq. in.}$$
$$= (62{\cdot}353 - 56{\cdot}571) \text{ sq. in. nearly}$$
$$= 5{\cdot}782 \text{ sq. in. nearly}$$

PROPOSITION XXI.

86. *To find the area of a sector of a circle, having given the length of its arc and the radius of the circle.*



Let OAB be a sector of the circle $ABCD$.

Let AB, the arc of the sector, measure l of any linear unit. Let OB. the radius of the circle $ABCD$, measure r of the same linear unit.

It is required to find the area of the sector in terms of l and r.

Draw OC at right angles to OA.

In any circle the areas of sectors are proportional to their arcs . Euc. VI. 33.

$\therefore$ in the circle $ABCD$, area of sector OAB : area of sector CAC $=$ arc AB : arc ABC

But $\angle AOC = 90^\circ$

$\therefore$ sector OAC is the fourth part of the circle $ABCD$

and arc ABC is the fourth part of the circumference $ABCD$

$$\therefore \text{area of sector } OAB : \frac{\text{area of circle } ABCD}{4}$$
$$= \text{arc } AB : \frac{\text{circumference of circle } ABCD}{4}$$

That is—

Area of sector $OAB : \frac{\pi r^2}{4} = l : \frac{2\pi r}{4}$ . . . . . . §§ 71, 69.

$\therefore$ area of sector $OAB = l \times \frac{\pi r^2}{4} \times \frac{4}{2\pi r}$ sq. units

$= \frac{1}{2}lr$ sq. units

Hence rule—

Multiply the number of any linear unit in the arc of a sector by the number of the same linear unit in the radius; then half the product will give the number of the corresponding square unit in the area.

Or briefly—

Area of sector = $\frac{1}{2}$ × arc × radius

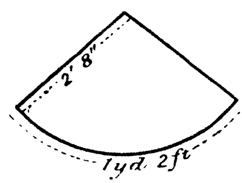
$$\mathbf{A} = \tfrac{1}{2}\mathbf{lr} \quad \text{. . . . . (i.)}$$

$$\text{Hence } \mathbf{l} = \frac{\mathbf{2A}}{\mathbf{r}} \quad \text{. . . . . (ii.)}$$

$$\text{and } \mathbf{r} = \frac{\mathbf{2A}}{\mathbf{l}} \quad \text{. . . . . (iii.)}$$

ILLUSTRATIVE EXAMPLES.

87. *Example* 1.—Find the area of a sector whose radius measures 2 ft. 8 in., and arc 1 yd. 2 ft.

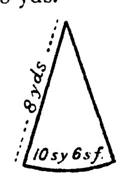


Area of sector $= \frac{1}{2}lr$ sq. ft. . § 86.

where $r = 2\frac{2}{3}$,
and $l = 5$;

$\therefore$ area of sector $= \frac{1}{2} \times 5 \times \frac{8}{3}$ sq. ft.
$= 6\frac{2}{3}$ sq. ft.
$= 6$ sq. ft. 96 sq. in.

Example 2.—Find the length of the arc of a sector whose arec measures 10 sq. yds. 6 sq. ft., and whose radius measures 8 yds.



Length of arc $= \frac{2A}{r}$ ft. . . . . § 86.

where $A = 96$,
and $r = 24$;

$\therefore$ length of arc $= \frac{2 \times 96}{24}$ ft.
$= 8$ ft.

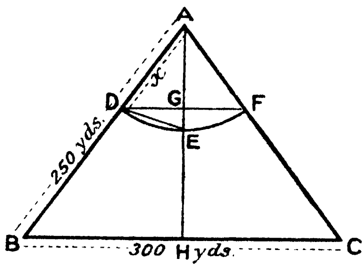
Example 3.—The area of a sector measures 4·8 sq. ch., and the length of its arc 60 lks.: find the radius.

Radius of sector $= \frac{2A}{l}$ ch. . . . . . § 86.

where $A = 4·8$,
and $l = 0·6$;

$\therefore$ radius of sector $= \frac{2 \times 4·8}{0·6}$ ch.
$= 16$ ch.

Example 4.—A field is in the form of an isosceles triangle, and measures 250 yds. along each of its equal sides, and 300 yds. along its base: what must be the length of a tether fixed at its apex and to a horse's nose to enable him to graze exactly one-fifth of it?



Let the $\triangle ABC$ represent the field.

If AD represent the length of tether required, the sector $ADEF$ will represent the area grazed.

Draw AH perpendicular to BC. Join DF, DE.

Let AD measure x yds.

It will be necessary to find (1) DF, (2) DE, both in terms of x, so that we can express the length of the arc DEF in terms of x, and then the area of the sector $ADEF$ in terms of x.

To find DF. By similar triangles—

$$DF : BC = AD : AB \quad . \; . \; . \; . \; . \; . \; \S\, 64.$$

$$\therefore DF = \frac{300x}{250} \text{ yds.}$$

$$= \frac{6x}{5} \text{ yds.}$$

To find DE. By similar triangles—

$$AG : AD = AH : AB \quad . \; . \; . \; . \; . \; . \; . \; . \; \S\, 64.$$

$$\text{But } AH = \sqrt{(250)^2 - (150)^2} \text{ yds.} \quad . \; . \; . \; . \; \S\, 64.$$

$$= \sqrt{400 \times 100} \text{ yds.}$$

$$= 200 \text{ yds.}$$

$$\therefore AG = \frac{200x}{250} \text{ yds.}$$

$$= \frac{4x}{5} \text{ yds.}$$

$$\therefore GE = AE - AG = \left(x - \frac{4x}{5}\right) \text{yds.}$$

$$= \frac{x}{5} \text{ yds.}$$

$$\text{Now, } DE^2 = DG^2 + GE^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; \S\, 16.$$

$$\therefore DE = \sqrt{\left(\frac{3x}{5}\right)^2 + \left(\frac{x}{5}\right)^2} \text{ yds.}$$

$$= \frac{x\sqrt{10}}{5} \text{ yds.}$$

$$\text{hence length of arc } DEF = \tfrac{1}{3}(8 \times DE - DF) \quad . \; . \; . \; . \; \S\, 81.$$

$$= \tfrac{1}{3}\left(8 \times \frac{x\sqrt{10}}{5} - \frac{6x}{5}\right) \text{yds.}$$

$$\text{and area of sector } ADEF = \tfrac{1}{2} \times AD \times \text{arc } DEF \quad . \; . \; . \; \S\, 86.$$

$$= \tfrac{1}{2} \times x \times \tfrac{1}{3}\left(\frac{8x\sqrt{10}}{5} - \frac{6x}{5}\right) \text{sq. yds.}$$

But area of sector $ADEF = \frac{1}{5} \times$ area of $\triangle ABC$

$$= \frac{1}{5} \times \frac{c}{4}\sqrt{4a^2 - c^2} \text{ sq. yds. . . § 24.}$$

where $c = 300$,
$a = 250$;

$$\therefore \text{this area} = \frac{1}{5} \times \frac{300}{4}\sqrt{(500)^2 - (300)^2} \text{ sq. yds.}$$
$$= \frac{1}{5} \times \frac{300}{4} \times 400 \text{ sq. yds.}$$
$$= 6000 \text{ sq. yds.}$$

$$\therefore \tfrac{1}{2} \times x \times \tfrac{1}{3}.\left(\frac{8x\sqrt{10}}{5} - \frac{6x}{5}\right) = 6000$$

$$\therefore \frac{x^2}{15}(4\sqrt{10} - 3) = 6000$$

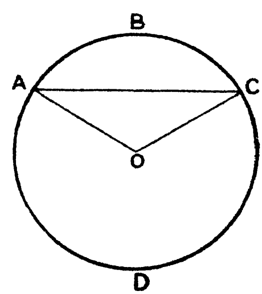
$$\therefore x^2 = \frac{90000}{4\sqrt{10} - 3}$$
$$= \frac{90000(4\sqrt{10} + 3)}{160 - 9}$$
$$= \frac{90000 \times 15{\cdot}6490 \ldots}{151}$$
$$= \frac{1408410}{151} \text{ nearly}$$
$$= 9327 \text{ nearly}$$
$$\therefore x = 96 \text{ nearly}$$

hence required length of tether = 96 yds. nearly

PROPOSITION XXII.

88. *To find the area of a segment of a circle.*

It may be seen from the figure that any chord, AC, divides a circle $ABCD$ into two segments—



(1) The segment ACB, which is less than a semicircle, and which we shall call the *minor* segment.

(2) The segment ACD, which is greater than a semicircle, and which we shall call the *major* segment.

It is also evident that—

(1) Area of minor segment ACB = area of sector $OABC$ − area of $\triangle OAC$.

(2) Area of major segment ACD = area of sector $OADC$ + area of $\triangle OAC$.

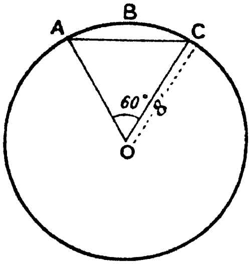
Hence rule—

To find the area of a segment of a circle, find the area of the sector which has the same arc, and then subtract or add the area of the triangle formed by the radii and chord, according as the segment is less or greater than a semicircle.

Or briefly—

Area of segment = area of sector ∓ area of triangle

ILLUSTRATIVE EXAMPLES.



89. *Example* 1.—The radius of a circle is 8 ft., and the angle of the sector is 60°: find the area of the segment. ($\pi = \frac{22}{7}$.)

Area of segment ACB = area of sector $OACB$ − area of $\triangle OAC$. . § 88.

Now, area of sector $OABC$ $= \frac{\alpha}{360} \times \pi r^2$ sq. ft. . . § 84.

where $\alpha = 60$,
and $r = 8$;

$\therefore$ area of sector $OABC = \frac{60}{360} \times \frac{22}{7} \times 8^2$ sq. ft.
$= \frac{704}{21}$ sq. ft.
$= 33{\cdot}523$ sq. ft. nearly

and area of $\triangle OAC = \frac{\sqrt{3}}{4}a^2$ sq. ft. . . . . . . § 21.

where $a = 8$;

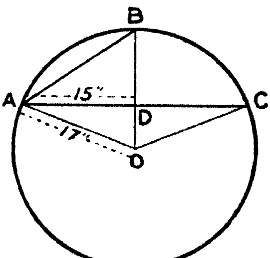
$\therefore$ area of $\triangle OAC = 27{\cdot}712$ sq. ft. nearly
hence, area of segment $= (33{\cdot}523 - 27{\cdot}712)$ sq. ft. nearly
$= 5{\cdot}811$ sq. ft. nearly

Example 2.—In a circle of radius 17 in. a segment is cut off by a chord whose length is 30 in.: find the area of the segment.

Segment ACB = sector $OABC - \triangle OAC$ . . § 88.
Now, $\triangle OAC = \frac{1}{2} \times AC \times OD$ sq. in.. . . . § 20.

where $AC = 30$,
and $OD = \sqrt{OA^2 - AD^2} = \sqrt{17^2 - 15^2} = 8$ . . . . . . § 16.

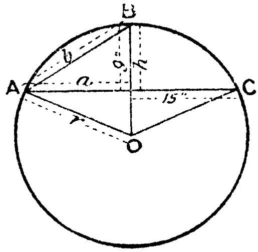
$\therefore \triangle OAC = \frac{1}{2} \times 30 \times 8$ sq. in.
$= 120$ sq. in.
and sector $OABC = \frac{1}{2} \times$ arc $\times$ radius . . . § 86.
$= \frac{1}{2} \times lr$ sq. in.



where $r = 17$,
and $l = \frac{8b - 2a}{3}$ . . . . § 81.
$= \frac{8 \times 17{\cdot}49\ldots - 30}{3}$ . . § 16.
$= 36{\cdot}64$ nearly;
$\therefore$ sector $OABC = \frac{1}{2} \times 36{\cdot}64 \times 17$ sq. in. nearly
$= 311{\cdot}44$ sq. in. nearly

Hence—
segment $ACB = (311{\cdot}44 - 120)$ sq. in. nearly
$= 191{\cdot}44$ sq. in. nearly

Example 3.—Find the area of a segment whose chord measures 30 in., and whose height measures 9 in.



$b^2 = 2hr$ . . . . § 77.

also $b^2 = a^2 + h^2$ . . . . § 16.

$\therefore 2hr = a^2 + h^2$

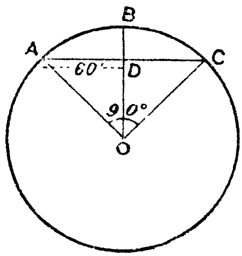
$$\therefore r = \frac{a^2 + h^2}{2h} = \frac{(15)^2 + (9)^2}{2 \times 9} = 17$$

that is—

radius of circle = 17 in.

Proceeding now as in previous example, we find that area of segment equals 191·44 sq. in. nearly.

Example 4.—The span of a circular arch of 90° is 120 ft.: find the area of the segment.



Segment ACB = sector $OABC - \triangle OAC$ § 88.

Now $\angle AOC = 90°$

$\therefore OD = AD = 60$ ft.

and $OA = AD\sqrt{2} = 60\sqrt{2}$ ft. . § 17.

Again, sector $OABC = \frac{1}{4}$ of circle

$\therefore$ area of sector $OABC = \frac{\pi r^2}{4}$ sq. ft. . § 71.

where $r = 60\sqrt{2}$,

and area of $\triangle OAC = \frac{1}{2}bh$ sq. ft. . § 20.

where $b = 120$,

and $h = 60$;

$$\therefore \text{area of segment} = \left(\frac{22}{7} \times \frac{60^2 \times 2}{4} - \frac{1}{2} \times 120 \times 60\right) \text{sq. ft.}$$
$$= (5657\tfrac{1}{7} - 3600) \text{ sq. ft.}$$
$$= 2057\tfrac{1}{7} \text{ sq. ft.}$$

Example 5.—Find the side of a square inscribed in a segment of a circle the chord of which is 12 in. and the height 4 in.

Let each side of the square measure x in.

Then in the figure we have—

$EF = x$ in.

$AB = 12$ in.

$CF = 4$ in.

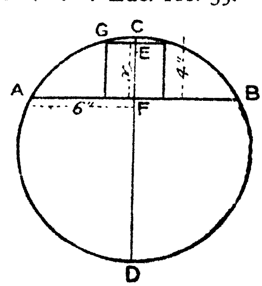
Now, $DF \times FC = AF^2$ . . . . Euc. III. 35.

$$\therefore DF = \frac{(6)^2}{4} \text{ in.}$$
$$= 9 \text{ in.}$$

$\therefore DE = DF + FE = (9 + x)$ in.

also $CE = (4 - x)$ in.

But $CE \times ED = GE^2$ Euc. III. 35



$$\therefore (4 - x)(9 + x) = \frac{x^2}{4}$$
$$\therefore 36 - 5x - x^2 = \frac{x^2}{4}$$
$$\therefore 144 - 20x - 4x^2 = x^2$$
$$x^2 + 4x = \frac{144}{5}$$

By solving this equation we find—

$$x = 3\cdot727 \text{ nearly}$$

Hence each side of the square measures

3·727 in. nearly

PROPOSITION XXIII.

90. *To find the area of a segment of a circle, having given the chord and height of the arc.*

Rule—

Add together one-fourth of the square of the number of any linear unit in the chord of the arc, and two-fifths of the square of the number of the same linear unit in the height; then multiply the square root of the sum by four-thirds of the number of the same linear unit in the height; and the product will give the number of the corresponding square unit in the area of the segment.

Or briefly—

$$\textbf{Area of segment} = \frac{4}{3}\,\textbf{height}\sqrt{\left(\frac{1}{4}\,\textbf{chord}^2 + \frac{2}{5}\,\textbf{height}^2\right)}$$

$$\mathbf{A} = \frac{4}{3}\mathrm{h}\sqrt{\left(\frac{1}{4}\mathrm{c}^2 + \frac{2}{5}\mathrm{h}^2\right)}$$

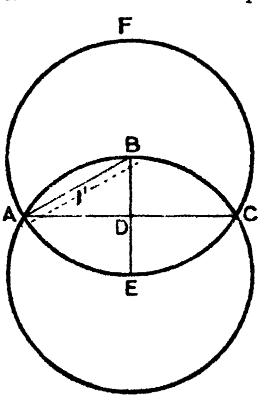
The proof of this formula is omitted, since it depends upon the higher mathematics.

The area of the segment as given by this formula is somewhat greater than it should be; but the error is very small, especially when the central angle of the arc is small.

Note.—For an alternative method, see § 89, Example 3.

ILLUSTRATIVE EXAMPLE.

91. *Example* 1.—Two equal circles intersect in such a way that the circumference of each passes through the centre of the other: if the radius of each circle measure 1 ft., find the area of the space common to both circles.



$$\left.\begin{array}{l}\text{Area of space com-}\\ \text{mon to both circles}\end{array}\right\} = 2 \times \text{area of segment } ACB$$

$$= 2 \times \tfrac{4}{3}h\sqrt{(\tfrac{1}{4}c^2 + \tfrac{2}{5}h^2)} \text{ sq. ft.} \quad . \quad . \quad . \quad . \quad \S\ 90.$$

where h = number of feet in BD,
and c = „ „ AC;

Now, $BD = \frac{1}{2}BE = \frac{1}{2}$ ft.

$\therefore h = \frac{1}{2}$

also $AD^2 = AB^2 - BD^2$. . § 16.

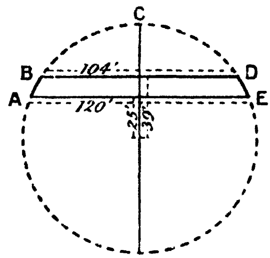
$\therefore \left(\frac{c}{2}\right)^2 = 1^2 - (\frac{1}{2})^2 = \frac{3}{4}$

$\therefore c = \sqrt{3}$

Hence—

$$\text{Required area} = 2 \times \tfrac{4}{3} \times \tfrac{1}{2} \times \sqrt{\{\tfrac{1}{4}(\sqrt{3})^2 + \tfrac{2}{5}(\tfrac{1}{2})^2\}} \text{ sq. ft.}$$
$$= \tfrac{4}{3}\sqrt{(\tfrac{3}{4} + \tfrac{1}{10})} \text{ sq. ft.}$$
$$= \tfrac{4}{3}\sqrt{(\tfrac{17}{20})} \text{ sq. ft.}$$
$$= \tfrac{4}{3} \times 0.922 \text{ sq. ft. nearly}$$
$$= 1.22 \text{ sq. ft. nearly}$$

Example 2.—If two parallel chords of a zone on the same side of the centre of the circle are 120 ft. and 104 ft.; and their distances from the centre 25 ft. and 39 ft.: find the area of the zone.



Area of zone $ABDE$ = area of segment ACE − area of segment BCD.

$$\left.\begin{array}{l}\text{Now, area of} \\ \text{segment } ACE\end{array}\right\} = \tfrac{4}{3}h\sqrt{(\tfrac{1}{4}c^2 + \tfrac{2}{5}h^2)} \text{ sq. ft. § 90.}$$

where $c = 120$,
and $h(50+h) = 60 \times 60$, that is, $h = 40$.

$$\therefore \left.\begin{array}{l}\text{area of seg-} \\ \text{ment } ACE\end{array}\right\} = \tfrac{4}{3} \times 40 \times \sqrt{(60)^2 + \tfrac{2}{5}(40)^2}\text{sq.ft.}$$
$$= 3472.7 \text{ sq. ft. nearly}$$

Also area of segment $BCD = \tfrac{4}{3}h\sqrt{(\tfrac{1}{4}c^2 + \tfrac{2}{5}h^2)}$ sq. ft. . . . . . § 90.
where $c = 104$, and $h(78+h) = 52 \times 52$, that is, $h = 26$.

$$\therefore \text{area of segment } BCD = \tfrac{4}{3} \times 26 \times \sqrt{(52)^2 + \tfrac{2}{5}(26)^2} \text{ sq. ft.}$$
$$= 1890.6 \text{ sq. ft. nearly}$$
$$\text{Hence area of zone } ABDE = (3472.7 - 1890.6) \text{ sq. ft. nearly}$$
$$= 1582.1 \text{ sq. ft. nearly}$$

Examples—XIV.

On Sectors.

$(\pi = \tfrac{22}{7}.)$

1. In a circle of radius 16 in., find the area of a sector whose angle measures 60°.

2. In a circle of radius 3 ft. 4 in., find the area of a sector whose angle measures 45°.

3. In a circle of radius 14 yds., find the area of a sector whose angle measures 6° 40′.

4. In a circle of radius 13 ch. 50 lks., find the area of a sector whose angle measures 13° 7′ 30″.

5. The area of a sector is 40 sq. ft.; the radius is 15 ft.: find the angle of the sector.

6. The area of a sector is 80 sq. ft.; the radius is 16 ft.: find the angle of the sector.

7. The area of a sector is 8 sq. ft.; the angle of the sector is 45°: find the radius of the sector.

8. The area of a sector is 36 sq. in.; the angle of the sector is 70°: find the radius of the sector.

9. Find the area of a sector whose radius measures 15 in., and arc 28 in.

10. Find the area of a sector whose radius measures 3 yds. 2 ft., and arc 4 yds. 1 ft.

11. Find the length of the arc of a sector whose area measures 15 sq. ft., and radius 6 ft.

12. Find the length of the arc of a sector whose area measures 3 sq. ft. 72 sq. in., and radius 4 yds. 2 ft.

13. The area of a sector measures 24 sq. in., and the length of its arc 8 in.: find its radius.

14. The area of a sector measures 2 sq. ft. 108 sq. in., and the length of its arc 5 ft. 6 in.: find its radius.

15. The area of a sector is 75 sq. in.; the area of the circle is 125 sq. in.: find the angle of the sector.

16. The chord of a sector is 6 in.; the radius is 5 in.: find the area.

17. The area of a sector is 240 sq. ft., and the area of the circle is 960 sq. ft.: find the length of the arc. $\pi = 3 \cdot 14159$.

18. In a circle whose area measures 1 ac., find the area of a sector whose angle measures $75°$.

On Segments.

($\pi = \frac{22}{7}$.)

19. The radius of a circle is 10 in., and the angle of the sector is $90°$: find the area of the segment.

20. Find the area of a segment of a circle of radius 2 ft. 6 in., if the chord of the segment subtends an angle of $120°$ at the centre of the circle.

21. Find the area of a segment of a circle whose chord measures $5\frac{1}{2}$ ft., and subtends an angle of $60°$ at the centre.

22. The chord of a segment is 8 ch. 40 lks., and it subtends an angle of $90°$ at the centre of the circle: find the area of the segment.

23. In a circle of radius $1\frac{1}{4}$ ch., find the area of a segment whose chord is equal to the radius of the circle.

24. In a circle of radius 10 in., find the area of the zone between two parallel chords drawn on the same side of the centre and subtending angles of $90°$ and $60°$ at the centre respectively.

25. In a circle of radius 8 ft., find the area of the zone between two parallel chords drawn on opposite sides of the centre and subtending angles of $90°$ and $120°$ respectively.

26. Find the area of a segment whose chord measures 8 yds., and whose height measures 2 yds. (Use § 90.)

27. The span of a circular arch of $60°$ is 140 ft.: find the area of the segment.

28. Find the area of the major segment cut off by a chord of 1 ch. from a circle whose radius is 1 ch. 30 lks.

Examination Questions—XIV.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Allahabad University: Matriculation.*

1. Three circles, each of radius 1 ft., touch each other: find the area of the curvilinear figure included between them. $\pi = 3 \cdot 14159$.

B. *Punjab University: Matriculation.*

2. A circular disc of cardboard 1 ft. in diameter is divided into six equal sectors, by pencil lines through the centre. In each sector there is described a circle touching the two bounding radii of the sector, and also the arc joining their ends at its middle point. If the circles are cut out from the six sectors, find the area of cardboard remaining.

3. The area of an equilateral triangle is 17,320·5 sq. ft. About each angular point, as centre, a circle is described with radius equal to half the length of a side of the triangle. Find the area of the space included between the three circles. ($\pi = 3{\cdot}14159.$)

C. *Calcutta University: Matriculation.*

4. Find the area of a sector of a circle with radius 35 ft., the angle of the sector being 15°.

5. A chord of a circle subtends an angle of 60° at the centre: if the length of the chord be 100, find the areas of the two segments into which the chord divides the circle.

D. *European Schools: Final. United Provinces.*

6. A field in the form of an equilateral triangle contains half an acre: what must be the length of a tether fixed at one of its angles and to a horse's nose, to enable him to graze exactly half of it?

E. *Madras Technical: Elementary.*

7. The length of the arc of a circle is 14 ft., and the radius is 10 ft.: find the area of the sector.

8. Find the area of a sector when the radius is 50 ft., and the length of arc 16 ft.

9. Find the area of a segment of a circle whose radius is 6 ft., the chord of whose arc is 8 ft.

F. *Roorkee Engineer: Entrance.*

10. If two circles be described on the bounding radii of a quadrant of a circle whose radius is 10 ft., as diameters, find the area of the figure common to both circles. $\pi = 3{\cdot}14159$.

11. Find the area of the space which is common to four equal circles which intersect each other, their centres being at the angular points of a square, and their radii equal to a side of the square.

12. The radius of a circle is 75; a zone of that circle has one of its parallel chords coinciding with the diameter, and the other equal to the radius: what is the area of the zone? $\pi = 3{\cdot}14159$.

13. Find the area of the zone of a circle contained between two parallel chords whose lengths are 96 and 60 in., and their distance apart 26 in.

14. The area of a segment of a circle which is less than a semicircle is 1 sq. in., the length of the arc of the segment is $2\frac{1}{2}$ in., and the perpendicular let fall from one end of the arc on to the diameter passing through the other end is $\frac{1}{2}$ in.: what is the radius of the circle?

15. A gentleman has a lawn in the shape of a circle; this he divides into quadrants, and in each he cuts out a circular walk 10 ft. in breadth, the outer edge of which touches the arc of the quadrant and its two radii. If the radius of the large circle be 100 ft., find the whole area of ground covered with grass.

16. Find the area of the zone of a circle whose diameter is 20, the parallel chords being 12 and 16, and both on same side of diameter.

17. Find the side of a square described in a segment of a circle, the chord of which is 20, and height 5 in.

18. If the centre of a circle whose diameter is 20 is in the circumference of another circle whose diameter is 40, what are the areas of the three included spaces?

19. The diameter of a circle is 25, and two parallel chords in it, on the same side of the centre, are 20 and 15: find the area of the zone contained by them.

20. The two parallel chords of a zone on opposite sides of the centre are

18 and 24, and their distances from the centre 12 and 9: find the area of the zone.

21. The lengths of the hour and minute hands of a clock are 10 and 13 in. respectively: find the difference of area of the sectors described by the hands between 11 hrs. 48 min. and 12 hrs. 14 min.

22. Find the area of the largest square that can be cut out of the segment of a circle whose chord is 16, and rise 4.

23. Given the chord 20 ft. and height 4 ft. of an arc of a circle, find the area of the segment.

24. Give formulæ for finding the area of a triangle, a regular polygon, and a sector of a circle.

G. *Roorkee Upper Subordinate: Entrance.*

25. Three equal circles intersect each other so that the circumference of each passes through the centres of the other two: find the area of the figure common to the three circles.

26. The chord of a sector is 6 in., the radius is 9 in.: find the area of the sector.

27. A field has the shape of an isosceles triangle whose base is 500 yds., and side 800 yds.: what length of rope shall I require to tether a horse at the apex of the triangle, so that he may graze over 1000 sq. yds.?

28. Two boys, amusing themselves at a game called snatch-apple in a room 30 ft. high, find that by standing 12 ft. from each other, the apple, which is suspended from the ceiling by a string, and is in a right line between them, will just touch each of their mouths: find the area of the sector formed by the apple and the string, the height of each boy's mouth from the floor being 3 ft.

29. The radius of a circle is $\sqrt{2}$ in.; two parallel straight lines are drawn in it, each an inch from the centre: find the area of the part of the circle between the straight lines.

30. The circumference of a circle is 11 ft.: find the length of the radius, and the area of the segment cut off by a chord equal to the radius.

31. Find the area of a sector, and of the segment of a circle whose chord is 24, and height 6.

32. The diameter of a rupee is $1\frac{1}{4}$ in.: if three of these coins be placed on a table so that the rim of each touches two others, it is required to find the area of the unoccupied space between them.

33. The radius of a circle is 25 ft.; two parallel chords are drawn, each equal to the radius: find the area of the zone between the chords. $\pi = 3{\cdot}14159$.

34. The radius of a circle is 12 ft.; two parallel chords are drawn on the same side of the centre, one subtending an angle of 60° at the centre, and the other an angle of 90°: find the area of the zone between the chords.

35. Find the area of a segment of a circle whose radius is 12, and chord 16.

36. What is the area of the sector of a circle whose arc of 24° measures 10 ft.?

H. *Roorkee Engineer: Final.*

37. Two equal circles of 1 in. radius are distant 2 in. from each other, and a cord passes tightly round them, crossing between them: find the length of the cord and the area enclosed by it. ($\pi = 3{\cdot}14159$.)

38. A circle whose diameter is 10 ft. passes through the extremities of a diameter of another, and bisects a radius at right angles: find the area of the part common to both. ($\pi = 3{\cdot}14159$.)

39. The radius of a circle is 15 ft.; two parallel chords are drawn on the same side of the centre, one subtending an angle of 60° at the centre, and the other an angle of 120°: find the area of the zone enclosed between the chords.

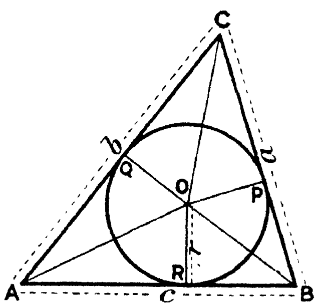
CHAPTER XV.

ON CIRCLES: INSCRIBED AND CIRCUMSCRIBED TO TRIANGLES.

PROPOSITION XXIV.

92. *To find the radius of the inscribed circle, having given the sides of the triangle.*

Let O be the centre of the circle PQR inscribed in the $\triangle ABC$.



Then, if P, Q, R be the points of contact between this circle and the triangle, OP, OQ, and OR will be radii of the inscribed circle, and they will be perpendicular to BC, CA, AB respectively . Euc. IV. 4.

Let OR, the radius of the circle PQR, measure r of any linear unit. Let BC, CA, AB measure a, b, c of the same linear unit respectively.

It is required to find r in terms of a, b, and c.

Join OA, OB, OC.

Now—

$$\left.\begin{array}{l}\text{Area of}\\ \triangle ABC\end{array}\right\} = \text{area of} \triangle BOC + \text{area of} \triangle COA + \text{area of} \triangle AOB$$

$$= \tfrac{1}{2} . OP . BC + \tfrac{1}{2} . OQ . CA + \tfrac{1}{2} . OR . AB \qquad . \ \S\ 20.$$

$$= (\tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc) \text{ sq. units}$$

$$= r\frac{a+b+c}{2} \text{ sq. units}$$

$$= rs \text{ sq. units} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \S\ 23.$$

$$\therefore \left.\begin{array}{l} r \text{ linear}\\ \text{units}\end{array}\right\} = \frac{\text{area of} \triangle ABC}{s \text{ linear units}}$$

$$r = \frac{\triangle}{s}$$

Hence rule—

The number of any square unit in the area of a triangle divided by the number of the corresponding linear unit in its semi-perimeter,

gives the number of the same linear unit in the radius of the inscribed circle.

Or briefly—

$$\textbf{Radius of inscribed circle} = \frac{\textbf{area of triangle}}{\textbf{semi-perimeter of triangle}}$$

$$r = \frac{\Delta}{s}$$

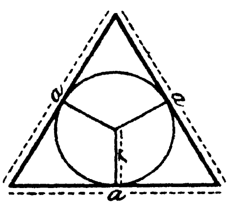
PARTICULAR CASE.

93. Equilateral triangle.

If a linear units denote each side of an equilateral triangle—

Then area of triangle $= \frac{a^2\sqrt{3}}{4}$ sq. units § 21.

and semi-perimeter of triangle $= \frac{3a}{2}$ linear units



Now, radius of circle inscribed in any triangle $= \frac{\text{area of triangle}}{\text{semi-perimeter of triangle}}$ § 92.

$\therefore$ radius of circle inscribed in equilat. triangle of side a $= \frac{a^2\sqrt{3}}{4} \times \frac{2}{3a}$ linear units

$= \frac{a}{2\sqrt{3}}$ linear units

Note.—Since the height of an equilateral triangle of side a is $\frac{a\sqrt{3}}{2}$ (see § 17), which may be written $\frac{a}{2\sqrt{3}} \times 3$, we see that the middle point of an equilateral triangle is situated at a distance from an angular point equal to $\frac{2}{3}$ of height of triangle.

ILLUSTRATIVE EXAMPLES.

94. *Example* 1.—The sides of a triangle measure 35, 44, 75 ft. respectively: find the radius of the inscribed circle.

Radius of inscribed circle $= \frac{\Delta}{s}$ ft. . . . . . § 92.

where $\Delta = \sqrt{s(s-35)(s-44)(s-75)}$ . . . . § 23.

and $s = \frac{35 + 44 + 75}{2} = 77,$

that is, $\Delta = \sqrt{77 \times 42 \times 33 \times 2},$
$= 462;$

$\therefore$ radius of inscribed circle $= \frac{462}{77}$ ft.
$= 6$ ft.

Example 2.—Find the circumference of the circle inscribed in an equilateral triangle whose side measures 9 yds. $\pi = \frac{22}{7}$.

Radius of inscribed circle $= \dfrac{a}{2\sqrt{3}}$ yds. . . . § 93.

where $a = 9$;

$\therefore$ radius of inscribed circle $= \dfrac{9}{2\sqrt{3}}$ yds.

$= \dfrac{3\sqrt{3}}{2}$ yds.

circumference of circle $= 2\pi r$ yds. . . . . . § 69.

where $r = \dfrac{3\sqrt{3}}{2}$

and $\pi = \frac{22}{7}$;

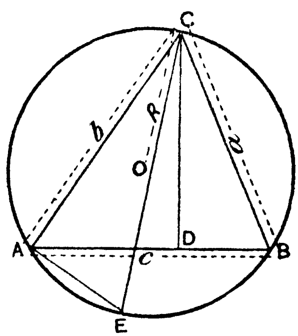
$\therefore$ circumference of circle $= 2 \times \frac{22}{7} \times \dfrac{3\sqrt{3}}{2}$ yds.

$= 16{\cdot}33 \ldots$ yds.

PROPOSITION XXV.

95. *To find the radius of the circumscribed circle, having given the sides of the triangle.*

Let O be the centre of the circle $AEBC$ circumscribed about the $\triangle ABC$.



Then OC will be a radius of the circumscribed circle.

Let OC, the radius of the circle $AEBC$, measure R of any linear unit. Let BC, CA, AB measure a, b, c of the same linear unit respectively.

It is required to find R in terms of a, b, and c.

Produce CO to meet the circumference of the circle $AEBC$ in E.

Join AE.

Draw CD perpendicular to AB.

Then, since triangles ACE, DCB are similar (Euc. III. 21 and III. 31)—

$\therefore BC : CD = EC : CA$ . . . . . § 64.

that is, $a : CD = 2R : b$

But $\frac{1}{2} . CD . AB =$ area of $\triangle ABC$ . . § 20.

that is, $\frac{1}{2} . CD . c = \triangle$

$$\text{or } CD = \frac{2\triangle}{c}$$

$$\therefore a : \frac{2\triangle}{c} = 2R : b$$

$$\text{or } R = \frac{abc}{4\triangle}$$

Hence rule—

The continued product of the numbers of the same linear unit in each of the three sides of a triangle, when divided by four times the number of the corresponding square unit in the area, gives the number of the same linear unit in the radius of the circumscribed circle.

Or briefly—

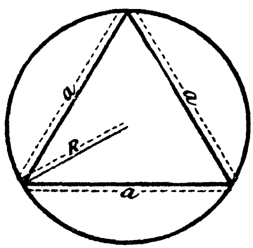
$$\textbf{Radius of circumscribed circle} = \frac{\textbf{product of sides of triangle}}{\textbf{4} \times \textbf{area of triangle}}$$

$$R = \frac{abc}{4\triangle}$$

PARTICULAR CASE.

96. Equilateral triangle.

If a linear units denote each side of an equilateral triangle—



Then area of triangle $= \frac{a^2\sqrt{3}}{4}$ sq. units § 21.

Now, radius of circle circumscribed about any triangle $= \frac{\text{product of sides}}{4 \times \text{area of triang.}}$. § 95.

$\therefore$ radius of circle circumscribed about an equilat. triangle of side a $= \dfrac{a^3}{4 \times \dfrac{a^2\sqrt{3}}{4}}$ linear units

$= \frac{a}{\sqrt{3}}$ linear units

ILLUSTRATIVE EXAMPLES.

97. *Example* 1.—The sides of a triangle measure 21, 72, 75 in. respectively: find the radius of the circumscribed circle.

Radius of circumscribed circle $= \frac{abc}{4\triangle}$ in. . . . § 95.

where $a = 21$, $b = 72$, $c = 75$,

and $\triangle = \sqrt{s(s-21)(s-72)(s-75)}$ . . . . § 23.

$= \sqrt{84 \times 63 \times 12 \times 9}$

$= 756$;

$\therefore$ radius of circumscribed circle $= \frac{21 \times 72 \times 75}{4 \times 756}$ in.

$= 37\frac{1}{2}$ in.

Example 2.—Find the area of the circle circumscribed about an equilateral triangle whose side measures 6 yds. $\pi = \frac{22}{7}$.

Radius of circumscribed circle $= \frac{a}{\sqrt{3}}$ yds. . . . § 96.

where $a = 6$;

$\therefore$ radius of circumscribed circle $= \dfrac{6}{\sqrt{3}}$ yds.
$= 2\sqrt{3}$ yds.

$\therefore$ area of circumscribed circle $= \pi r^2$ sq. yds. . . . § 71.

where $r = 2\sqrt{3}$,
and $\pi = \frac{22}{7}$;

$\therefore$ area of circumscribed circle $= \frac{22}{7} \times (2\sqrt{3})^2$ sq. yds.
$= \dfrac{22 \times 12}{7}$ sq. yds.
$= 37\frac{5}{7}$ sq. yds.

Examples—XV.

1. Find the radii of the inscribed and circumscribed circles to a triangle whose sides measure 24, 30, and 18 ft. respectively.

2. Find the radii of the inscribed and circumscribed circles to a triangle whose sides measure 36, 29, and 25 in. respectively.

3. Find the area of the circle (1) inscribed in, (2) circumscribed about, an equilateral triangle whose side measures 1 ft. 3 in. $\pi = \frac{22}{7}$.

4. Find the circumference of the circle (1) inscribed in, (2) circumscribed about, an equilateral triangle whose side measures 2 yds. 2 ft. 9 in. $\pi = \frac{22}{7}$.

Examination Questions—XV.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Allahabad University: Matriculation.*

1. Prove the formula for determining the radius of a circle inscribed in a triangle whose sides are given.

The radius of a circle inscribed in an equilateral triangle is 10 ft.: find the area of the triangle.

2. Find the diameter of the circle round a triangle whose sides are 123, 122, and 49.

B. *Punjab University: Matriculation.*

3. The three sides of a triangle being given, find the radius of the circle described about the triangle.

Apply your result to the triangle whose sides are 20, 48, and 52 ft.

4. The sides of a triangle are $2\frac{1}{4}$, 3, and $3\frac{3}{4}$ ft.: find in inches the radii of the inscribed and circumscribed circles.

C. *Punjab University: School Middle.*

5. Two sides of a triangle are 85 and 154 ft. respectively, and the perimeter is 324 ft.: find the diameter of the circle described round the triangle.

6. The sides of a triangle are 100, 156, and 160 yds. respectively: find the diameter of the circumscribing circle.

7. The sides of a triangle are 6, 7, and 9 ft. respectively: what is the diameter of the circle described round the triangle?

D. *European Schools: Final. United Provinces.*

8. Given a circle of radius 1 ft., find to three places of decimals the side of an equilateral triangle inscribed in it.

9. Find the diameter of the circle circumscribing a triangle, the sides of which are 68, 285, and 293 ft. respectively.

10. Calculate the side of an equilateral triangle inscribed in a circle whose radius is R.

E. *Madras Technical: Elementary.*

11. Find the length of the side of an equilateral triangle inscribed in a circle 8 in. diameter.

F. *Roorkee Engineer: Entrance.*

12. Two sides of a triangle containing an obtuse angle are equal to 10 and 14 in. respectively, and the perpendicular on the third side from the vertex is equal to 7 in.: find the diameter of the circumscribing circle.

13. The three sides of a triangle inscribed in a circle are 120, 160, and 180 ft. respectively: find the difference between the area of the circle and the area of the triangle.

14. Find the area, in square chains, of the circle inscribed in a triangle, of which the sides are 372, 350, and 320 yds. respectively.

G. *Roorkee Upper Subordinate: Monthly.*

15. The sides of a triangle are respectively 26, 28, and 30 in.: required the diameter of the circumscribing circle.

H. *Roorkee Upper Subordinate: Entrance.*

16. Find the side of an equilateral triangle inscribed in a circle whose radius is 10 in.

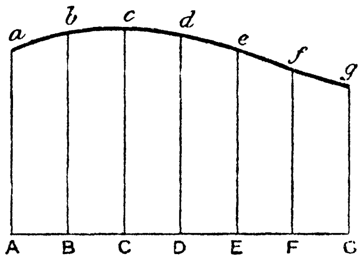
I. *Additional Examination Question.—XV.*

17. Show by computation that a side of the equilateral triangle together with a side of the square, both inscribed in the same circle, are equal to half the circumference of that circle, nearly. (Punjab University: Matriculation.)

CHAPTER XVI.

ON SIMPSON'S RULE.

98. CONSIDER the figure bounded by the curve adg, the straight line AG, and the two straight lines Aa and Gg at right angles to AG.



Let AG be divided into any number of equal parts AB, BC, CD . . .

At the points of division draw Bb, Cc . . . perpendiculars to AG, to meet the curve in b, c, d . . .

The perpendiculars Aa, Bb, Cc . . . are called the *ordinates* of the curve.

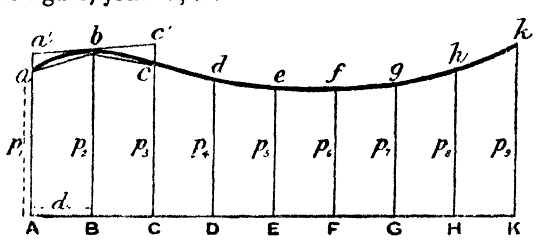
The length of AB is called the *common distance* between the ordinates.

PROPOSITION XXVI.

99. *To find approximately the area of a figure of which one of the bounding lines is a curve, having given the lengths of an odd number of ordinates and their common distance.*

Let the lengths of the ordinates measure $p_1, p_2, \ldots p_9$, and their common distance d of the same linear unit.

In the figure, join ab, bc.



Through b draw a tangent to the curve, meeting the ordinates Aa and Cc produced in a' and c'.

We shall call the portions into which the figure is divided by the ordinates *pieces*.

$$\begin{aligned} \text{Area of first piece } AabB &> \text{area of trapezoid } aB \\ &> \tfrac{1}{2}\,.\,AB\,.\,(Aa + Bb) \qquad .\ \S\ 39. \\ &> \tfrac{1}{2}d(p_1 + p_2) \text{ sq. units} \\ \therefore\ 2 \times \text{area of first piece} &> d(p_1 + p_2) \text{ sq. units} \end{aligned}$$

Similarly—

$$\begin{aligned} 2 \times \text{area of second piece} &> d(p_2 + p_3) \text{ sq. units} \\ \therefore\ 2 \times (\text{area of first piece} &+ \text{area of second piece}) \\ i.e.\ 2 \times \text{area of } AacC &> d(p_1 + p_3 + 2p_2) \text{ sq. units} \end{aligned}$$

Again, area of first and second pieces—

$$\begin{aligned} i.e. \text{ area of } AacC &< \text{area of trapezoid } a'C \\ &< AB(Aa' + Cc') \\ &< AB \times 2Bb \\ &< d \times 2p_2 \text{ sq. units} \end{aligned}$$

Thus we have obtained two results, one of which is somewhat too small to be taken for the area of $AacC$, and the other is somewhat too large.

By combining these two results, the errors will, to a large extent, balance one another. Hence, for all practical purposes, we may write—

$$3 \times \text{area of } AacC = d(p_1 + p_3 + 4p_2) \text{ sq. units}$$

We now proceed to the third and fourth pieces, which together make up the figure $CceE$.

We need only substitute the letter p_3 for p_1, p_4 for p_2, p_5 for p_3, in the expression which we have obtained for the area of $AacC$.

Thus we have—

$$3 \times \text{area of } CceE = d(p_3 + p_5 + 4p_4) \text{ sq. units}$$

Similarly, we find—

$$\begin{aligned} 3 \times \text{area of } EegG &= d(p_5 + p_7 + 4p_6) \text{ sq. units} \\ \text{and } 3 \times \quad ,, \quad GgkK &= d(p_7 + p_9 + 4p_8) \text{ sq. units} \end{aligned}$$

Hence—

$$\text{Total area} = \frac{d}{3}\{p_1 + p_9 + 2(p_3 + p_5 + p_7) + 4(p_2 + p_4 + p_6 + p_8)\} \text{ sq. units}$$

Hence rule—

Add together the first and last ordinates, twice the sum of the remaining odd ordinates, and four times the sum of the even ordinates; multiply the result by one-third the common distance

Or briefly—

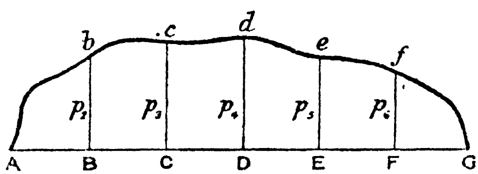
$$\textbf{Area} = \frac{\textbf{common distance}}{3}\{\textbf{first ordinate} + \textbf{last ordinate} \\ + 2 \times \textbf{sum of remaining odd ordinates} \\ + 4 \times \textbf{sum of even ordinates}\}$$

$$\mathbf{A} = \frac{\mathbf{d}}{\mathbf{3}}\{\mathbf{p}_1 + \mathbf{p}_{2n+1} + \mathbf{2}(\mathbf{p}_3 + \mathbf{p}_5 + \ldots + \mathbf{p}_{2n-1}) \\ + \mathbf{4}(\mathbf{p}_2 + \mathbf{p}_4 + \ldots + \mathbf{p}_{2n})\}$$

where $2n + 1$ = number of ordinates

It is evident that by using a greater number of ordinates we shall obtain, by this rule, a more accurate result. Moreover, the accuracy depends upon the regularity of the curve.

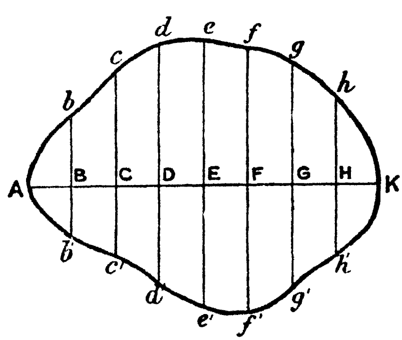
100. If the area be bounded by a curve and a straight line,



as in the figure, the same rule applies. In this case the first and last ordinates vanish, so that the formula becomes—

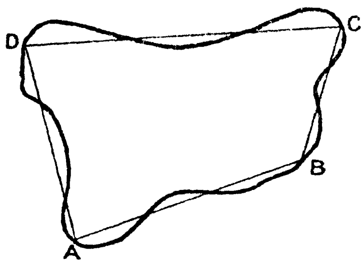
$$A = \frac{d}{3}\{2(p_3 + p_5) + 4(p_2 + p_4 + p_6)\}$$

101. The same rule also holds good when the area is bounded by a closed curve, as in the figure.



In this case the total lengths bb', cc', ... **are taken as the ordinates.**

102. When a figure, such as $ABCD$, is bounded by an irregular closed curve, it will be found advisable to describe a rectilineal figure coinciding as nearly as possible with the curve. By Simpson's rule we can then determine the areas of those portions of the curvilinear figure which lie outside the rectilineal figure, and also the areas of those portions of the rectilineal figure which lie outside the curvilinear figure. By adding the former areas to, and subtracting the latter areas from, the area of the rectilineal figure, we obtain the area of the curvilinear figure.



ILLUSTRATIVE EXAMPLES.

103. *Example* 1.—The ordinates of a curve measure 5, 6, 8, 7, 4, 3, 2 ft. respectively, and their common distance is 1 ft.: find the area of the figure.

$$\text{Area} = \frac{d}{3}\{p_1 + p_7 + 2(p_3 + p_5) + 4(p_2 + p_4 + p_6)\} \text{ sq. ft.} \quad . \ \S\ 99.$$

where $p_1 = 5$, $p_2 = 6$, $p_3 = 8$, $p_4 = 7$, $p_5 = 4$, $p_6 = 3$, $p_7 = 2$, and $d = 1$;

$$\begin{aligned}\therefore \text{Area} &= \tfrac{1}{3}\{5 + 2 + 2(8 + 4) + 4(6 + 7 + 3)\} \text{ sq. ft.} \\ &= \tfrac{1}{3}\{7 + 24 + 64\} \text{ sq. ft.} \\ &= 31\tfrac{2}{3} \text{ sq. ft.}\end{aligned}$$

Example 2.—The ordinates of a curve are 0, 1·25, 2, 2·5, 2·75, 3·5, 2·5, 1, 0 ft. respectively, and the common distance is $\frac{3}{4}$ ft.: find the area.

$$\text{Area} = \frac{d}{3}\{p_1 + p_9 + 2(p_3 + p_5 + p_7) + 4(p_2 + p_4 + p_6 + p_8)\} \text{ sq. ft.} \quad \S\ 99.$$

where $p_1 = 0, p_2 = 1{\cdot}25, p_3 = 2, p_4 = 2{\cdot}5, p_5 = 2{\cdot}75, p_6 = 3{\cdot}5, p_7 = 2{\cdot}5, p_8 = 1, p_9 = 0$, and $d = 0{\cdot}75$;

$$\begin{aligned}\therefore \text{Area} &= \tfrac{0{\cdot}75}{3}\{2(2 + 2{\cdot}75 + 2{\cdot}5) + 4(1{\cdot}25 + 2{\cdot}5 + 3{\cdot}5 + 1)\} \text{ sq. ft.} \\ &= 0{\cdot}25\{14{\cdot}5 + 33\} \text{ sq. ft.} \\ &= 11{\cdot}875 \text{ sq. ft.}\end{aligned}$$

Examples—XVI.

Find the areas of the following curvilinear figures by Simpson's Rule:—

1. Ordinates, 4, 7, 8, 10, 7, 6, 3 ft.; common distance, 1 ft.
2. Ordinates, 1, 3, 5, 6, 7, 9, 8, 4, 2 ft.; common distance, 1 ft.
3. Ordinates, 9, 13, 17, 20, 22, 14, 8 ft.; common distance, 2 ft.
4. Ordinates, 0, 1·5, 2·25, 3, 2·75, 1·25, 0 ft.; common distance, 1 ft.
5. Ordinates, 7, 10, 12, 14, 22, 22, 18, 15, 9 ft.; common distance, 1 ft. 6 in.

6. Ordinates, 1·428, 1·536, 1·690, 1·746, 1·768, 1·786, 1·804 ft.; common distance 0·75 ft.

7. Ordinates, 0, 1·54, 1·86, 2, 2·6, 2·98, 3·4, 4, 2·8 ft.; common distance, 1 ft.

8. Ordinates, 4·638, 4·872, 4·986, 5·004, 5·264, 5·186, 4·888, 4·684, 4·228 ft.; common distance, 1 ft.

9. Ordinates, 1·3268, 1·4394, 1·5868, 1·6264, 1·7644, 1·6896, 1·5482, 1·4466, 1·4088 ft.; common distance, 1 ft.

10. Ordinates, 0·04, 0·3688, 0·4, 0·57, 0·482, 0·3348, 0·266, 0·18, 0 ft.; common distance, 0·05 ft.

11. Ordinates, 0, $1\frac{3}{4}$, $2\frac{1}{3}$, $3\frac{3}{8}$, $2\frac{1}{3}$, $1\frac{2}{3}$, 0 ft.; common distance, $\frac{1}{2}$ ft.

12. Ordinates, 0, 14, 18, 20, 22, 28, 30, 26, 24, 20, 16, 14, 12 ft.; base line, 18 ft.

Examination Questions—XVI.

A. *Madras Technical: Elementary.*

1. Apply Simpson's Rule to find in square feet the area of a field having the following dimensions: ordinates, 0, 20, 32, 36, 32, 20, 0 ft.; the common distance being 20 ft.

2. Apply Simpson's Rule to find the area of a plot of land having the following dimensions: ordinates, 2, 7, 18, 38, 70 ft.; common distance, 33 ft.

B. *Madras Technical: Intermediate.*

3. Apply Simpson's Rule to find the area in square feet of a figure having the following ordinates: 2, 7, 9, 15, 21, 30, 12 ft.; common distance, 33 ft.

C. *Roorkee Engineer: Entrance.*

4. Find the area of a quadrant by Simpson's Rule by dividing it into 10 spaces of equal breadth. Lastly, show the error in the result. ($\pi = 3·1416$.)

D. *Roorkee Engineer: Monthly.*

5. One side of a field, *AB*, is 80 ft. long; two other sides, *AC* and *BD*, at right angles to it, are respectively 16 and 60 ft.; the fourth side is curved, and the ordinates to it from *AB* (at intervals of 10 ft. from *A*) are 18, 24, 36, 21, 28, 40, and 50: what is the area of the field?

6. Thirteen equidistant ordinates are measured to a curve at 100 ft. intervals on a chain line: find the area between the end ordinates, the curve, and the chain line. The ordinates are: 50, 60, 80, 90, 30, 50, 60, 80, 70, 90, 100, 120, 130.

E. *Roorkee Engineer: Final.*

7. The length of a line is 584 ft., and at equal distances along it the following offsets were taken to an irregularly curved fence, viz. 93, 84, 72, 68, 43, 54, 37, 29, and 23 ft.: find the area included between the extreme offsets, the fence, and the chain line.

8. Apply Simpson's Rule to find the area of a section, the heights of which above the railway level, at intervals of 30 ft., are 2, 10, 15, 20, 30, 25, 17·5, 10, 3 ft.

9. Describe what is meant by "Simpson's Rule." Under what circumstances is it applicable?

F. *Roorkee Upper Subordinate: Monthly.*

10. Find by Simpson's Rule the area of a curvilinear figure whose ordinates are 9, 11, 13, 12, 10, 13, 15, 17, 14, 12, 7 ft.; base = 73 ft.

G. *Roorkee Upper Subordinate: Final.*

11. By Simpson's Rule find the area of a figure having the following dimensions: ordinates, 5, 15, 37, 77, and 141; common distance, 2 ft.

CHAPTER XVII.

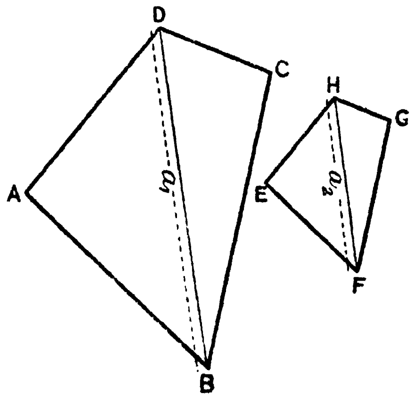
ON SIMILAR FIGURES: THEIR AREAS.

PROPOSITION XXVII.

104. *Having given the lengths of corresponding lines drawn in two similar figures, and the area of one of these figures, to find the area of the other figure.*

Let $ABCD$ and $EFGH$ be two similar figures.

Let the corresponding lines DB and HF measure a_1 and a_2 of



the same linear unit respectively. Let the area of the figure $EFGH$ measure A_2 of any square unit.

It is required to find the area of the figure $ABCD$ in terms of a_1, a_2, and A_2.

Now, it is proved in Euclid that the areas of similar figures are proportional to the squares of any corresponding lines that may be drawn in them, whether the figures be rectilineal or curvilinear.

$\therefore$ area of figure $ABCD$: area of figure $EFGH = DB^2 : HF^2$

That is—

$$\text{Area of figure } ABCD : A_2 = a_1^2 : a_2^2$$

Hence rule—

The area of a figure is found by taking its ratio to the known area of a similar figure, and equating it to the ratio of the squares of known corresponding lengths.

Or briefly—

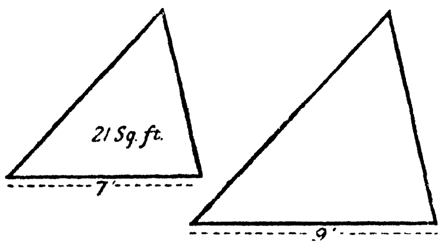
Area of first figure : area of second figure = ratio of the squares of corresponding lengths in first figure and second figure.

$$\mathbf{A}_1 : \mathbf{A}_2 = \mathbf{a}_1^2 : \mathbf{a}_2^2 \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{Hence } \mathbf{a}_1 : \mathbf{a}_2 = \sqrt{\mathbf{A}_1} : \sqrt{\mathbf{A}_2} \quad . \quad . \quad . \quad \text{(ii.)}$$

ILLUSTRATIVE EXAMPLES.

105. *Example* 1.—In two similar triangles the bases are 7 ft. and 9 ft. respectively. If the area of the first is 21 sq. ft., what is the area of the second?



$$\text{Area of the second} : A_1 \text{ sq. ft.} = (a_2)^2 : (a_1)^2 \quad . \quad . \quad \S\ 104.$$

where $A_1 = 21$,
$a_1 = 7$,
$a_2 = 9$;

$$\therefore \text{ area of second} = 21 \times \tfrac{81}{49} \text{ sq. ft.}$$
$$= \tfrac{243}{7} \text{ sq. ft.}$$
$$= 34\tfrac{5}{7} \text{ sq. ft.}$$

Example 2.—The plan of a field containing 2400 sq. yds. is drawn to a scale of 1 in. to 30 ft.: find the area of the plan.

$$\text{Area of plan} : A_2 \text{ sq. yds.} = (a_1)^2 : (a_2)^2 \quad . \quad . \quad \S\ 104.$$

where $A_2 = 2400$,
$a_1 = 1$,
$a_2 = 30 \times 12 = 360$;

$$\therefore \text{ area of plan} = 2400 \times \frac{1}{(360)^2} \text{ sq. yds.}$$
$$= \frac{2400 \times 9 \times 144}{360 \times 360} \text{ sq. in.}$$
$$= 24 \text{ sq. in.}$$

Example 3.—If in a plan every square inch of surface represents an area of ten acres, find the scale to which it has been drawn.

$$(a_1) \text{ in.} : (a_2) \text{ in.} = \sqrt{A_1} : \sqrt{A_2} \quad . \quad . \quad . \quad . \quad \S\ 104.$$

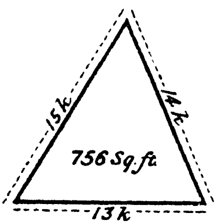
where $A_1 = 1$,
$A_2 = 4840 \times 9 \times 144 \times 10$;

$$\therefore (a_1) \text{ in.} : (a_2) \text{ in.} = \sqrt{1} : \sqrt{4840 \times 9 \times 144 \times 10}$$
$$= 1 : 7920$$

Thus the scale is—

1 in. to 7920 in.
or 1 in. to $\frac{1}{8}$ mi.

Example 4.—The sides of a triangle are in the proportion of the numbers 13, 14, and 15. Its area is 756 sq. ft.: find the three sides.



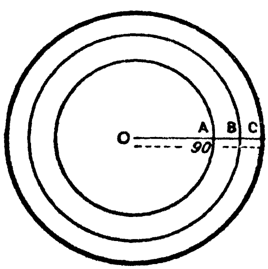
$$\text{First side} : 13 \text{ ft.} = \sqrt{A_1} : \sqrt{A_2} \quad \S\ 104.$$

where $A_1 = 756$,
and $A_2 = \sqrt{s(s-13)(s-14)(s-15)}$ § 23.
$= \sqrt{21 \times 8 \times 7 \times 6}$
$= 84$;

$$\therefore \text{first side} = 13 \times \sqrt{\tfrac{756}{84}} \text{ ft.}$$
$$= 13 \times 3 \text{ ft.}$$
$$= 39 \text{ ft.}$$

Hence second and third sides are 42 ft. and 45 ft. respectively.

Example 5.—A circle of 90 ft. radius is divided into three parts by two concentric circles: find the radii of these circles, so that the three parts may be of equal area.



Let OC be the radius of the given circle, and let OB and OA be the radii of the two concentric circles respectively, by which it is divided into three equal parts.

Let OA measure r_1 ft., and OB r_2 ft.

Then $r_2^2 : r_1^2 = 2 : 1$ . . . . . . § 104.
and $(90)^2 : r_1^2 = 3 : 1$ . . . . . . § 104.

$$\text{Hence } r_1 = \frac{90}{\sqrt{3}} = 30\sqrt{3} = 51{\cdot}96 \ldots$$
$$\text{and } r_2 = r_1\sqrt{2} = 30\sqrt{6} = 73{\cdot}48 \ldots$$

Thus the radii are seen to measure $51{\cdot}96 \ldots$ ft. and $73{\cdot}48 \ldots$ ft. respectively.

Examples—XVII.

1. In two similar triangles the bases are 8 in. and 11 in. respectively: if the area of the first is 128 sq. yds., what is the area of the second?

2. If the area of one triangle is nine times the area of a similar triangle, and if the base of the first triangle measure 9 in., find the base of the second triangle.

3. A plan of a field is drawn to a scale of 1 in. to 8 ft.: if the field contain 640 sq. yds., how many square inches will the plan occupy?

4. A plan of a field is drawn to the scale of 1 in. to 9 ft. 2 in. : if the field contain $672\frac{2}{9}$ sq. yds., how many square inches will the plan occupy?

5. In the construction of a plan, a square foot of surface represents an area of $1\frac{3}{5}$ ac. : determine the scale.

6. If in a plan every square inch of surface represent an area of 90 ac., find the scale to which it has been drawn.

7. The sides of a triangle are in the proportion of the numbers 13, 14, 15 ; its area is 336 sq. ft. : find the three sides.

8. The sides of a triangle are in the proportion of the numbers 3, 4, 5 ; its area is 96 sq. ft. : find the three sides.

9. The sides of a rectangle are in the proportion of the numbers 8 and 9, and the area is 32 sq. ft. : find the sides.

10. The sides of a rectangle are in the proportion of the numbers 12 and 13, and the area is 4 sq. ft. 48 sq. in. : find the sides.

11. A triangle is divided into two equal parts by a straight line parallel to the base : if the base measure 18 ft., find the length of the straight line.

12. A triangle is divided into three equal parts by two straight lines parallel to the base : find the lengths of these two straight lines if the base measure 80 in.

13. A circle of 100 in. radius is divided into two parts by a concentric circle : find the radius of this concentric circle so that the two parts may be of equal area.

14. A circle of 5 ft. radius is divided into three parts by two concentric circles : find the radii of these circles so that the three parts may be of equal area.

Examination Questions—XVII.

A. *Allahabad University: Matriculation.*

1. A circle of 120 ft. radius is divided into three parts by two concentric circles : find the radii of these circles so that the three parts may be of equal area.

B. *Punjab University: Matriculation.*

2. The ratio of the similar sides of two similar triangles is 13 : 17 : find the ratio of their areas.

C. *Calcutta University: Matriculation.*

3. What will be the size of the map of a district which contains 676 sq. mi., on a scale of 4 mi. to the inch?

D. *European Schools: Final. United Provinces.*

4. The sides of a triangle are 20, 13, and 21 ft. ; a straight line is drawn from the middle point of the side of 20 ft. across the triangle parallel to the longest side : find the area of the two parts into which the triangle is divided.

5. Of two concentric circles the area of the smaller is half that of the larger : find the radius of the larger circle, that of the smaller being 4 ft.

E. *Madras Technical: Intermediate.*

6. Find the scale to which a plan is drawn, 1 sq. in. representing 10 ac.

7. What is the scale to which a plan is drawn if 1 sq. in. represents 1 ac. ?

F. *Madras Technical: Elementary.*

8. The sides of a triangle are 39, 52, and 65 ft. respectively : find the sides of a similar triangle of nine times its area.

9. One side of a field measuring 28 ac. and 9 cents is 17 ch. : what is the area of a similar field whose corresponding side measures 27 ch. ?

G. *Roorkee Engineer : Entrance.*

10. If from a right-angled triangle whose base is 12 and perpendicular height 16 ft., a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 sq. ft., what are the sides of this triangle?

11. The radius of a circle is 20 in. ; it is required to draw three concentric circles in such a manner that the whole area may be divided into four equal parts : find the radii.

12. Out of a circular disc of metal 35 equal holes are punched ; the weight of metal thus punched out is to the weight of the perforated disc as 45 : 67 : compare the diameters of the disc and of the holes, given that the area of a circle varies as the square of the diameter.

H. *Roorkee Upper Subordinate : Entrance.*

13. The radius of a circle is 18 in. : find the radius of another circle of one-fifth the area.

14. The area of a rectangular plot, whose length is equal to twice its breadth, is 3528 yds.: if a gravel path 4 ft. wide goes along one of the diagonals of the plot, find the area of the path.

15. Find the height of a triangle which shall be similar to, and contain five times as much as, another 50 ft. in altitude.

16. Find the dimensions in feet of a trapezoid which shall contain 100 sq. yds., similar to one whose parallel sides are 8 and 10, and perpendicular distance between them 4.

17. On a map drawn to the scale of $\frac{1}{10000}$, the sides of a rectangular field are 0·65 and 0·72 in. : find the area of the field in acres, and the length of the diagonal in yards.

18. One side of a triangle is 20 ft. : divide the triangle into five equal parts by straight lines parallel to one of the other sides, and find the distances, from the vertex of the points of division of the given side.

19. A circle whose area is 314·16 sq. in. is to be divided into four equal portions by concentric circles : find their diameters. ($\pi = 3·1416$.)

20. The sides of a triangle are 532, 427, and 389 ft. : find the length of a line, parallel to the longest side, that will divide the triangle into two equal parts.

21. A drawing is copied to a scale one-half as large again as the original scale : in what ratio is its surface augmented?

I. *Roorkee Engineer : Final.*

22. The sides of a triangular field are 3501, 3604, 3605 ft. respectively : find the length of a line drawn parallel to the longest side which will divide the field into two equal parts.

J. *Roorkee Upper Subordinate : Monthly.*

23. A circular hole is to be cut in a circular plate so that the weight may be reduced one-third : find the diameter of the hole.

24. A board is 12 in. wide at one end, and 9 in. at the other end, and its length is 8 ft. : how far from the broad end must it be sawn across so as to divide the plank into two equal portions?

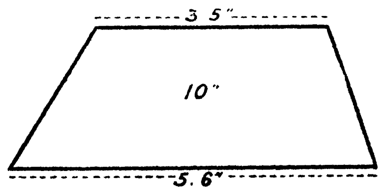
25. Find the dimensions of a triangle similar to one whose dimensions are 50, 60, and 80 ft., but which shall contain three times the quantity.

CHAPTER XVIII.

HINTS ON SOLVING PROBLEMS—COLLECTION OF FORMULÆ.

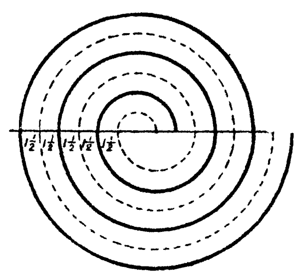
HINTS ON SOLVING PROBLEMS.

106. I. DRAW a figure, whenever possible, to illustrate the question, and let the figure show the various measurements that may be given.



Example.—The area of a trapezoid is 10 sq. in., and its parallel sides measure 3·5 in. and 5·6 in. respectively: find the perpendicular distance between the parallel sides.

II. Do not draw your figure in three dimensions when a section in two dimensions will serve the purpose equally well.

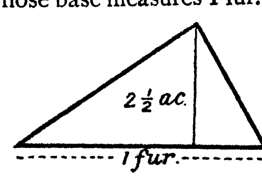


Example.—A perfectly flexible rope of 3 in. in diameter is coiled closely upon the deck of a ship, and there are three complete coils: find the length of the rope.

III. Decide upon the unit in terms of which you want your answer to appear, and adopt it throughout. If both areas and lengths are involved, let your unit of area correspond to your unit of length.

Example.—How many chains are there in the height of a triangle whose area measures $2\frac{1}{2}$ ac., and whose base measures 1 fur.?

Height of triangle $= \dfrac{2A}{b}$ ch. § 20.



where $A = 2\frac{1}{2} \times 10 = 25$
and $b = 10$

$\therefore$ height of triangle $= \frac{50}{10}$ ch.
$= 5$ „

Notice that the answer is to appear in chains. For this reason we reduce the $2\frac{1}{2}$ ac. to square chains by multiplying by 10, and the furlong to linear chains by multiplying by 10.

IV. Use contracted multiplication of decimals and contracted division of decimals when only an approximate result is required.

Example.—Find the product of 2·49832684 by 12·345976 correct to four decimal places.

2·4983	2684
12·3459	76
24·9832	6
4·9966	5
·7494	9
999	3
124	9
22	4
1	6
	1
30·8442	

Notice that, in this method, we multiply by the several digits of the multiplier in order from left to right instead of from right to left, as is usually the case. Figures in the multiplicand are crossed out as soon as they become superfluous by giving products which do not affect the accuracy of the required answer.

Example.—Divide 93·62135079 by 7·5083624, the result to be correct to two decimal places.

75083	93621	1246
	75083	
	18538	
	15016	
	3522	
	3003	
	519	
	450	

Required result is 12·46

We see, by inspection, that there will be *two* integral figures in the quotient.

Hence the result must contain *four* significant figures, since it is to be correct to two decimal places.

To allow for this we need only retain the first *five* figures of the divisor and the same number of the dividend.

The decimal points are now omitted.

Figures in the divisor are crossed out as soon as they become superfluous.

V. Avoid intermediate arithmetical calculations, so that your answer may, whenever possible, depend upon the simplification of a complex fraction.

Example.—Find the cost of paving a rectangular enclosure 450 ft. long and 390 ft. broad, with stones 3 ft. 9 in. by 3 ft. 3 in., at Rs. 5 8 annas a hundred.

$$\left.\begin{array}{r}\text{Area of en-}\\\text{closure}\end{array}\right\} = 450 \times 390 \text{ sq. ft.}$$

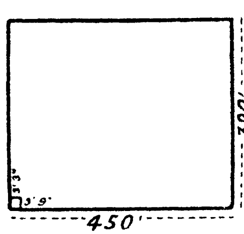
$$\left.\begin{array}{r}\text{area of each}\\\text{stone}\end{array}\right\} = 3\tfrac{3}{4} \times 3\tfrac{1}{4} \text{ sq. ft.}$$

$$\therefore \left.\begin{array}{r}\text{number of}\\\text{stones reqd.}\end{array}\right\} = \frac{450 \times 390}{3\frac{3}{4} \times 3\frac{1}{4}}$$

$$\therefore \text{cost} = \text{Rs.}\frac{450 \times 390 \times 5\frac{1}{2}}{3\frac{3}{4} \times 3\frac{1}{4} \times 100}$$

$$= \text{Rs.}\frac{\overset{3}{\cancel{450}} \times \overset{3}{\cancel{390}} \times 11 \times \overset{2}{\cancel{4}} \times 4}{\cancel{15} \times \cancel{13} \times \cancel{100} \times \cancel{2}}$$

$$= \text{Rs.}792$$



VI. It is often advisable not to substitute their values for such terms as π, $\sqrt{2}$, $\sqrt{3}$, . . . until the end is reached, particularly if there is a chance of their cancelling.

Example.—Compare the areas of an equilateral triangle and a regular hexagon, the length of side in each being equal.

Let a side of each measure a inches.

Then—

$$\text{Area of equilateral triangle} = \frac{a^2\sqrt{3}}{4} \text{ sq. in.}$$

$$\text{and area of hexagon} = \frac{3a^2\sqrt{3}}{2} \text{ sq. in.}$$

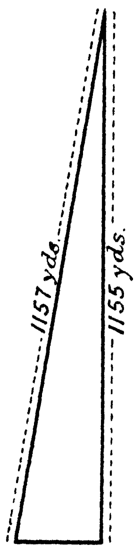
$$\therefore \text{area of triangle : area of hexagon} = \frac{a^2\sqrt{3}}{4} : \frac{3a^2\sqrt{3}}{2}$$

$$= 1 : 6$$

VII. The square root of the product of several numbers may often be found by inspection, by resolving the numbers into duplicate factors.

Example.—If $a = 143$, $b = 220$, $c = 231$, find the value of $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$.

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{297 \times 154 \times 77 \times 66}\\ &= \sqrt{11 \times 27 \times 2 \times 77 \times 77 \times 6 \times 11}\\ &= \sqrt{11^2 \times 77^2 \times 2^2 \times 9^2}\\ &= 11 \times 77 \times 2 \times 9\\ &= 15246\end{aligned}$$



VIII. Remember that—

$$\sqrt{h^2 - b^2} = \sqrt{(h-b)(h+b)}$$
and
$$\sqrt{h^2 - p^2} = \sqrt{(h-p)(h+p)}$$

Example.—Find the base of a right-angled triangle whose hypotenuse and perpendicular measure 1157 yds. and 1155 yds. respectively.

$$\begin{aligned}\text{Base} &= \sqrt{h^2 - p^2}\text{ yds.}\\ &= \sqrt{(h-p)(h+p)}\text{ yds.}\end{aligned}$$

where $h = 1157$
and $p = 1155$

$$\begin{aligned}\therefore \text{base} &= \sqrt{(1157 - 1155)(1157 + 1155)}\text{ yds.}\\ &= \sqrt{2 \times 2312}\text{ yds.}\\ &= \sqrt{2^2 \times 34^2}\text{ yds.}\\ &= 2 \times 34\text{ yds.}\\ &= 68\text{ yds.}\end{aligned}$$

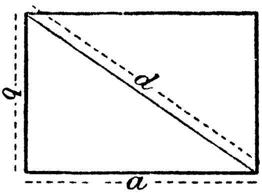
IX. Discard an answer which is not borne out by the geometry of the figure, or which, for other reasons, is obviously absurd.

For example, it may be required to find the length of the chord of an arc of a circle when the height of the arc and the diameter of the circle are given. An answer which makes the chord greater than the diameter of the circle is evidently wrong.

Or, again, it is often easy to make a rough guess at the answer if the figure be fairly accurately drawn to scale. This is particularly easy when the required answer is a length. A result that appears wide of the mark may safely be discarded.

COLLECTION OF FORMULÆ.

107. 1. **Rectangles.**



(i.) $A = ab$

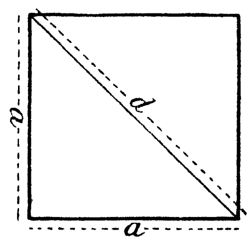
(ii.) $d = \sqrt{a^2 + b^2}$

where A = area; a = length; b = breadth; d = diagonal.

(iii) $a = \sqrt{d^2 - b^2}$

(iv) $b = \sqrt{d^2 - a^2}$

2. **Squares.**

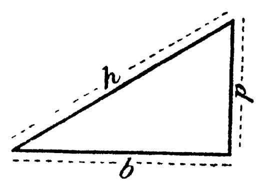


(i.) $A = a^2$

(ii.) $d = a\sqrt{2}$

where A = area; a = side; d = diagonal.

3. **Right-angled triangles.**



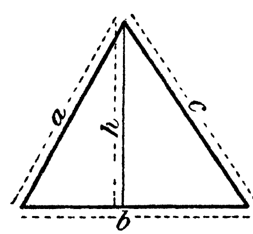
(i.) $h = \sqrt{b^2 + p^2}$

(ii.) $b = \sqrt{(h - p)(h + p)}$

(iii.) $p = \sqrt{(h - b)(h + b)}$

where h = hypotenuse; b = base; p = perpendicular.

4. **Triangles.**



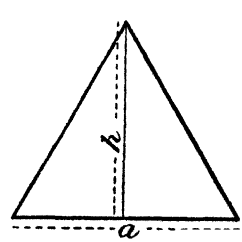
(i.) $A = \frac{1}{2}bh$

where A = area; b = base; h = height.

(ii.) $A = \sqrt{s(s - a)(s - b)(s - c)}$

where A = area; a, b, and c are the three sides; $s = \dfrac{a + b + c}{2}$.

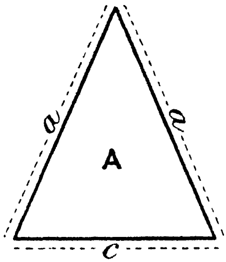
5. **Equilateral triangles.**



(i.) $h = \dfrac{a\sqrt{3}}{2}$

(ii.) $A = \dfrac{\sqrt{3}}{4} \times a^2$

where h = height; a = side; A = area.



6. Isosceles triangles.

$$A = \frac{c}{4}\sqrt{4a^2 - c^2}$$

where A = area ; a = side ; c = base.

7. Parallelograms.

(i.) $A = bh$

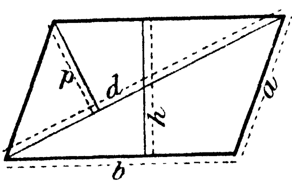
where A = area; b = base ; h = height.

(ii.) $A = dp$

where A = area ; d = diagonal ; p = offset of diagonal.

(iii.) $A = 2\sqrt{s(s-a)(s-b)(s-d)}$

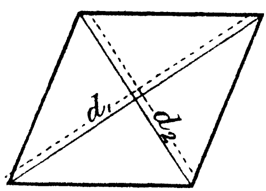
where A = area ; d = diagonal ; a and b are two adjacent sides ; $s = \frac{a+b+d}{2}$.

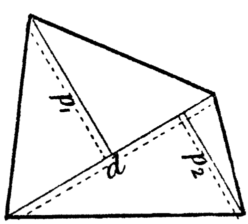


8. Rhombus.

$$A = \tfrac{1}{2} \, . \, d_1 d_2$$

where A = area ; d_1 and d_2 are the two diagonals.

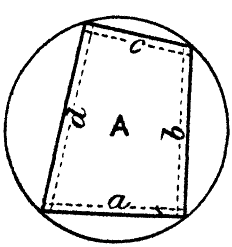




9. Quadrilaterals.

$$A = \tfrac{1}{2} \, . \, d \, . \, (p_1 + p_2)$$

where A = area ; d = diagonal ; p_1 and p_2 are the offsets of the diagonal.

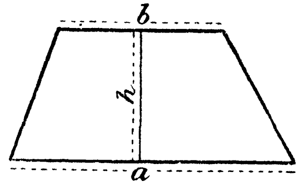


10. Quadrilaterals inscribed in circles.

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where A = area ; a, b, c, d are the sides ; $s = \frac{a+b+c+d}{2}$.

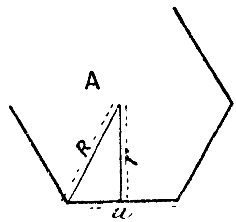
11. **Trapezoids.**



$$A = \tfrac{1}{2}(a + b)h$$

where A = area; a and b are the parallel sides; h = the perpendicular distance between the parallel sides.

12. **Regular polygons.**



(i.) $A = \frac{n}{2} \times ar$

(ii.) $A = \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$

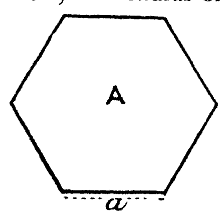
(iii.) $A = a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}$

(iv.) $A = r^2 \times n \tan \frac{180^\circ}{n}$

(v.) $A = R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n}$

where A = area; n = number of sides; a = side; r = radius of inscribed circle; R = radius of circumscribed circle.

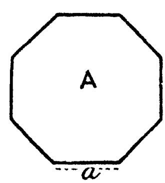
13. **Regular hexagons.**



$$A = \frac{3a^2\sqrt{3}}{2}$$

where A = area; a = side.

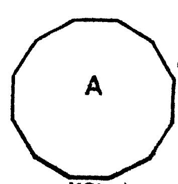
14. **Regular octagons.**



$$A = 2a^2(1 + \sqrt{2})$$

where A = area; a = side.

15. **Regular dodecagons.**



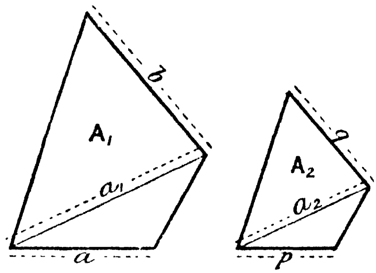
$$A = 6a^2\sqrt{\tfrac{7}{4} + \sqrt{3}}$$

where A = area; a = side.

16. Similar figures.

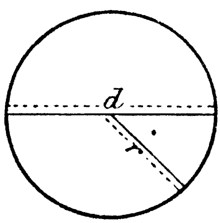
$$\text{(i.)} \quad a : b = p : q$$

where a and b are lengths in one figure, corresponding to p and q respectively in the other.



$$\text{(ii.)} \quad A_1 : A_2 = (a_1)^2 : (a_2)^2$$

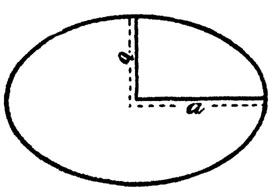
where A_1 and A_2 are the areas of the two figures; a_1 and a_2 are corresponding lengths, one in each figure.



17. Circles.

$$\text{(i.)} \quad C = \pi d$$
$$\text{(ii.)} \quad A = \pi r^2$$

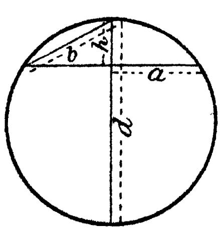
where C = circumference; d = diameter; A = area; r = radius.



18. Ellipses.

$$A = \pi ab$$

where A = area;
a = semi-major axis;
b = semi-minor axis.

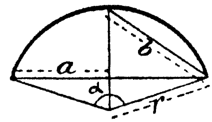


19. Chords of circles.

$$\text{(i.)} \quad a = \sqrt{h(d - h)}$$
$$\text{(ii.)} \quad b = \sqrt{dh}$$

where a = semi-chord of the arc; b = chord of the semi-arc; h = height of the arc; d = diameter of the circle.

20. Arcs of circles.

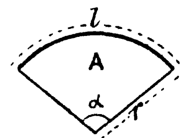


(i.) $l = \dfrac{\alpha}{360} \times 2\pi r$

(ii.) $l = \dfrac{8b - 2a}{3}$

where l = length of the arc; α° = central angle of the arc; r = radius of the circle; a = semi-chord of the arc; b = chord of the semi-arc.

21. Sectors of circles.

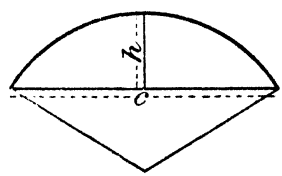


(i.) $A = \dfrac{\alpha}{360} \times \pi r^2$

(ii.) $A = \frac{1}{2} \times lr$

where A = area; α° = angle of the sector; l = length of the arc of the sector; r = radius of the circle.

22. Segments of circles.

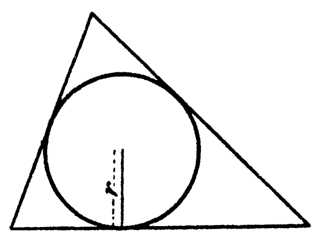


(i.) Segment = sector − triangle

(ii.) $A = \frac{4}{3}h\sqrt{(\frac{1}{4}c^2 + \frac{2}{5}h^2)}$

where A = area; h = height of the segment; c = chord of the segment.

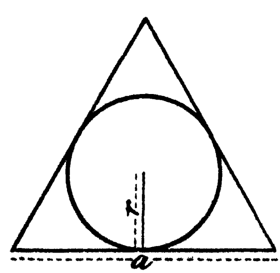
23. Circles inscribed in triangles.



$$r = \frac{\Delta}{s}$$

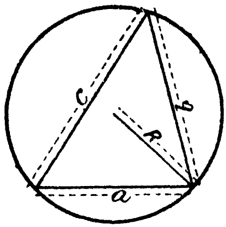
where r = radius of the inscribed circle; Δ = area of the triangle; s = semi-perimeter of the triangle.

24. Circles inscribed in equilateral triangles.



$$r = \frac{a}{2\sqrt{3}}$$

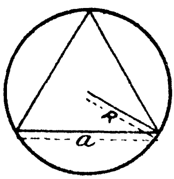
where r = radius of the inscribed circle; a = side of the triangle.



25. Circles circumscribed about triangles.

$$R = \frac{abc}{4\Delta}$$

where R = radius of the circumscribing circle; Δ = area of the triangle; a, b, c are the three sides of the triangle.



26. Circles circumscribed about equilateral triangles.

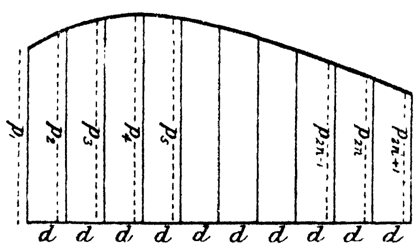
$$R = \frac{a}{\sqrt{3}}$$

where R = radius of the circumscribing circle; a = side of triangle.

27. Simpson's Rule.

$$A = \frac{d}{3}\{p_1 + p_{2n+1} + 2(p_3 + p_5 + \ldots + p_{2n-1}) \\ + 4(p_2 + p_4 + \ldots + p_{2n})\}$$

where A = area; d = common distance; $2n$ = number of equal



parts into which the base-line is divided; $p_1, p_2 \ldots p_{2n+1}$ are the ordinates taken in order.

ANSWERS

Examples—I. A.

1. $746\frac{1}{2}$ yds. **2.** $15193\frac{1}{2}$ ft. **3.** $30682\frac{1}{2}$ ft. **4.** 480 ch.
5. 560 sq. po. **6.** $23413\frac{1}{2}$ sq. yds. **7.** 900,000 sq. lks.
8. 18 mi. 5 fur. 23 po. 4 yds. $2\frac{1}{2}$ ft. **9.** 7·38629 ac.
10. 20 ac. 2 ro. 6 po. $5\frac{1}{2}$ sq. yds. 8 sq. ft. **11.** 140. **12.** $2117\frac{1}{2}$.
13. 16 yds. **14.** 605 sq. yds.

Examples—I. B.

15. 112 girahs. **16.** 10 lathas. **17.** 1750 haths. **18.** 5600 girahs.
19. 6800 biswansi. **20.** 13 bighas 6 biswas 1 biswansi. **21.** 640 biswansi.
22. 94 biswas.

Examples—II. A.

1. 143 sq. ft. **2.** 513 sq. ft. **3.** 1679 sq. ft.
4. 4 sq. ft. 51 sq. in. **5.** 9 sq. ft. 54 sq. in. **6.** 16 sq. ft. 8 sq. in.
7. 3 sq. yds. 3 sq. ft. 90 sq. in. **8.** 12 sq. yds. 2 sq. ft.
9. 18 sq. yds. 5 sq. ft. 82 sq. in. **10.** 46·8 ac. **11.** 10·944 ac.
12. 13·67174 ac. **13.** 2 ac. 0 ro. 23·2 sq. po.
14. 6 ac. 1 ro. 4·0128 sq. po. **15.** 6 ac. 2 ro. 21·76 sq. po. **16.** 2 yds. 9 in.
17. 121 yds. **18.** 121 yds. **19.** 15 ch.
20. 9 yds. 2 in. **21.** 15 ch. **22.** 88 yds.
23. 6 ch. 35 lks. **24.** 5 fur. 13 po. **25.** 87 sq. yds. 1 sq. ft.
26. 58 sq. yds. 7 sq. ft. **27.** 160 sq. yds. 4 sq. ft. **28.** 266 sq. yds. 7 sq. ft.
29. 22·5 ac. **30.** 17·66241 ac. **31.** 4·67856 ac.
32. 14·25636 ac. **33.** 18 ft. **34.** 3·5 ch.
35. 23·4 ch. **36.** 6 ch. 52 lks. **37.** Rs.1764.
38. Rs.59. **39.** 102 sq. yds. 6 sq. ft. **40.** 84 yds.
41. $341\frac{1}{3}$ sq. ch. **42.** 1 sq. yd. 8 sq. ft. 52 sq. in. **43.** 13 ch.
44. Rs.380. **45.** 4 yds. 1 ft. 8 in. **46.** 10 ch. ; 40 ch.
47. 256 sq. yds. **48.** $41\frac{1}{4}$ yds. **49.** Rs.$424\frac{2}{3}$.
50. Rs.196 5 annas 4 pies.

Examples—II. B.

51. 35 bighas. **52.** 9·12 bighas. **53.** 2 bighas.
54. 6 rasi. **55.** 2 rasi 15 lathas. **56.** $159\frac{97}{128}$ yds.

Examination Questions—II.

1. 540 sq. ft. **2.** 23,880 trees. **3.** £37 16*s.* $4\frac{1}{2}$*d.* **4.** 168 students.
5. £10 10*s.* **6.** $1782\frac{1}{4}$ sq. ft. **7.** 6 ft. **8.** £16 3*s.* $6\frac{2}{3}$*d.*
9. 70·5 po. **10.** 10 yds. 2 ft. 5·07 in. **11.** 24 yds. ; 36 yds.
12. Length, 31 ft. ; breadth, 21 ft. ; height, 13 ft. **13.** Rs.43 13 annas 6 pies.
14. 3·84 ac. ; 2·16 ac. **15.** 8·653 ft. nearly. **16.** 3·84 ac. **17.** 9 ft.

18. $9\frac{3}{4}$ in. **19.** 12 ft. **20.** $13\frac{1}{2}$ ft. **21.** 13.

22. $3\cdot\dot{3}$. **23.** 486 planks. **24.** $13\frac{1}{3}$ min. **25.** £33.

26. 280 sq. ft **27.** 820 in. **28.** 18 ft.

29. $26\frac{1}{9}$ yds. ; Rs.9 12 annas 8 pies. **30.** 6·49 . . . yds.

31. $180\frac{1}{9}$ yds. **32.** $40\frac{2}{9}$ sq. ft. **33.** 6400 trees **34.** 8·866 ft.

35. $946\frac{2}{3}$ ft. **36.** 33. **37.** £49,227 5*s*. $5\frac{5}{11}$*d*. **38.** 18·5017 ft.

39. 32 yds. **40.** 98 yds. 2 ft. **41.** 81.

Examples—III.

1. 3 ft. 7′. **2.** 13 ft. 8′ 6″. **3.** 9 ft. 6′ 9″. **4.** 10 ft. 7′ 4″.

5. 8 ft. 9′ 1″ 6‴. **6.** 6 sq. ft. 2′. **7.** 10 sq. ft. 4′ 4″. **8.** 8 sq. ft. 8′ 7″.

9. 13 cub. ft. 2′ 3″ 2‴. **10.** 12 cub. ft. 5′ 0″ 11‴. **11.** 16 cub. ft. 6′ 8″ 3‴.

12. 18 cub. ft. 9′ 5″ 6‴ 6⁗. **13.** 7 sq. ft. 2′. **14.** 14 sq. ft.

15. 41 sq. ft. 4′. **16.** 31 sq. ft. 3′ 8″. **17.** 48 sq. ft. 10′ 8″.

18. 0 sq. ft. 3′ 10″ 6‴. **19.** 0 sq. ft. 5′ 4″ 3‴ 4⁗. **20.** 37 sq. ft. 8′ 4″ 8‴.

21. 78 sq. ft. 6′ 11″ 6‴ 8⁗. **22.** 45 sq. ft. 0′ 2″ 6‴ 8⁗.

Examples—IV. A.

1. 78 in. **2.** 125 yds. **3.** 176 yds. 2 ft. **4.** 10 yds. 10 in.

5. 1 mi. 5 fur. 37 po. **6.** 11 ch. 90 lks. **7.** 27 ch. 50 lks. **8.** 39·73 ch.

9. 88·022 lks. **10.** 3·3 in. **11.** 7 in. **12.** 3 mi.

13. 6 ch. 72 lks. **14.** 8 mi. 2 fur. **15.** 210 yds. ; 168 yds.

16. 12·72 . . . in. **17.** 9 ft. **18.** 85 in. **19.** 4 ft. 3 in.

20. 18·38 . . . in. **21.** 2 ft. 6 in. **22.** 7 ft. 4 in. **23.** 4·47 ch.

24. Rs.724 4 annas. **25.** 995 yds. 2 ft. **26.** 49·4 ft. nearly. **27.** 243 mi. nearly.

28. 44 ft. 4 in. **29.** 62·6 ft. nearly. **30.** 14·4 ch. ; 10·8 ch.

Examples—IV. B.

31. 25 rasi. **32.** 85 lathas. **33.** 157 girahs. **34.** 229 haths.

35. 365 guz. **36.** 433 rasi. **37.** 9 lathas. **38.** 28 rasi.

39. 60 girahs. **40.** 105 haths. **41.** 84 guz. **42.** 120 rasi.

Examination Questions—IV.

1. 15 ft. ; 36 ft. **2.** 11·618 . . ft. **3.** 80 ft.

4. 46·861 . . . mi. **5.** 1. **7.** 44 ft.

8. 1·41421 mi. ; 640 ac. **9.** 120 ft. **10.** 4·2426 . . . yds.

11. 16 ft. **13.** 40·588 . . . ft. **14.** 12·6194 . . . ft.

15. 2 sq. ft. **16.** 99·701 . . . po. **17.** $16\sqrt{2}$ in.

18. 637 ft. ; 245 ft. **19.** 12 ft. **20.** 12,637 ft. ; 12,012 ft.

21. 8 ft. **22.** 5000 sq. yds. **23.** 562,500 sq. ft.

24. $20\sqrt{2}$ ft. **25.** Rs.549 13 annas 6·2 . . . pies.

Examples—V. A.

1. 184 sq. ft. **2.** 7 sq. yds. $8\frac{1}{2}$ sq. ft. **3.** 8 sq. yds. 7 sq. ft. $91\frac{1}{2}$ sq. in.

4. 6·39375 ac. **5.** 9 ft. **6.** 11 yds. 2 ft. 8 in.

7. 81 ch. $92\frac{64}{83}$ lks. **8.** 48·7179 ch. **9.** 12 sq. ft. 6 sq. in.

10. 4·61 sq. yds. nearly. **11.** 1·07226 ac. **12.** 4 sq. yds. 2 sq. ft. 138 sq. in.

13. 125·14 . . . sq. ft. **14.** 6 sq. yds. 5 sq. ft. 112 sq. in. nearly.

15. 17·736 . . . sq. ch. **16.** 90 sq. ft. **17.** 2310 sq. lks.

18. 21,294 sq. in. **19.** 86,700 sq. ft. **20.** 747,054 sq. lks.

21. 12·24 sq. ch. **22.** 50 in. **23.** 12 ch. 32 lks.

24. 1216·3 . . . sq. lks. **25.** Rs. 3·69 nearly a month. **26.** 7 ft. 5 in.

27. 22,638 trees. **28.** 1676·74 sq. ft. **29.** 6·928 . . . yds.

30. 77·2 . . yds. **31.** 18 : $\sqrt{3}$.

Examples—V. B.

32. $97\frac{1}{2}$ bighas. **33.** 12 bighas 13·8 biswas. **34.** 13 bighas 6 biswas 16 biswansi.
35. 64 lathas. **36.** 8 lathas. **37.** $98\frac{2}{11}$ lathas.
38. 3·897 bighas. **39.** 16 biswas 16 biswansi.

Examination Questions—V.

1. 15 ft. **2.** (*a*) Area = the amount of surface in any figure ; (*b*) 28 ft.
3. 60 ft. **4.** 22·7 . . . in. **5.** 14·4 . . . ft. **6.** 216·50 . . . ft.
7. 12 ft. **8.** 84 sq. ft. **9.** 486 sq. in.
10. 2772 sq. ft. **11.** 21·217 . . . sq. ft. **12.** 169·705.
13. 30 ; $6\frac{1}{4}$. **14.** 14 ch. 15·3 . . . lks. **15.** 24 ft.
16. £12 18*s*. **17.** 3570 sq. yds. **18.** 43·301 . . . sq. ft.
19. 28·9 . . . ft. **20.** 39·6862 . . . sq. ft. **21.** 317 ft. nearly.
22. 10 ac. 1 ro. 33 per. nearly. **23.** $4537\frac{1}{2}$ sq. yds ; $2722\frac{1}{2}$ sq. yds.
24. 3 ac. 2 ro. 9·088 po. **25.** 72·74 . . . ft. **26.** 58·8 . . . po.
27. 45 ft. ; 630 sq. ft. ; 540 sq. ft. **28.** 277·47 . . . yds., or 1200·24 . . . yds.
29. 22·4 . . . yds. ; 25·84 . . . yds. ; 24 yds. **30.** 0·537 . . . ac.
31. 56·3 . . . sq. lks. **32.** 1 ac. 1 ro. $24\frac{112}{121}$ po.
33. 6·928 . . . yds. **34.** 620,964 yds. nearly.
35. 34·64 . . . ft. ; 519·6 . . . sq. ft. **36.** 4·2 . . . yds. **37.** £4.
38. $119\frac{1}{21}$ sq. yds. **39.** 2400 ; 2600 ; 1800 ; 3200 sq. ft.
40. 15 ac. 0 ro. 37·91 po. **41.** 1 ro. 26·6 po. **42.** 709 ft. nearly.
43. 3468 sq. ft. **44.** £1218 19*s*. $3\frac{1}{9}$*d*. **45.** 67 ft. nearly.
46. 9841·251 sq. ft. **47.** 125·9 . . . lks. **48.** £10 4*s*. 9*d*.
49. £2 15*s*. **50.** 305·8 . . . sq. ft. ; 369·6 . . . sq. ft.
51. 57·19 ft. **52.** 192·837 sq. lks. **53.** Rs.407 5 annas.
54. 93,944·755 sq. yds. **55.** 21 ac. 1 ro. 12·7 po. **56.** 233 sq. yds. 3 sq. ft.
57. 128·9 . . . ft. **58.** £4. **59.** 202 ft. nearly.
60. 240 yds. **61.** 120 sq. ft. **62.** 14·941 in. **63.** 1·5 ft.

Examples—VI. A.

1. 312 sq. ft. **2.** 52 sq. yds. 8 sq. ft. **3.** 196·6512 sq. ch.
4. 8 ft. **5.** 1 yd. 2 ft. **6.** 2·631 ch.
7. 12 sq. ft. 108 sq. in. **8.** 11·58144 ac. **9.** 2 yds.
10. 106·82 . . . in. **11.** Rs.89 2 annas. **12.** 3511 sq.in. nearly.
13. 66 ch.

Examples—VI. B.

14. 320 bighas. **15.** 16 bighas 9 biswas 12 biswansi.
16. 56 bighas 8 biswas 16 biswansi. **17.** $12\frac{1}{33}$ lathas.
18. 1 bigha 4 biswas. **19.** $4\frac{7}{8}$ biswas.

Examination Questions—VI.

1. 5 ft. ; $4\frac{4}{5}$ ft. **2.** 3456 ; 60. **3.** 2 ac. 47·25 cents.
4. 8000 sq. ft. **5.** 8 ft. **6.** 256 sq. in. ; $8\sqrt{5}$ in.
7. 729·15 ft. ; 329·15 ft. **8.** 1200 sq. yds. ; 144·2 . . . yds ; 33·28 yds.
9. 10,296 sq. ft. ; 125 ft. ; 82·368 ft. **10.** 69·71 . . . ft. ; 627·39 . . . sq. ft.
11. 346·427712 ; 20·0016. **12.** 1054 ft. ; 625 ft. ; 566·63 ft.
13. 1350 sq. ft. ; $37\frac{1}{2}$ ft. ; 36 ft. **14.** 384 sq. ft.

Examples—VII. A.

1. 21,300 sq. ft. **2.** 102 sq. ft. **3.** 44·6424 sq. ch.
4. 93·9107 sq. ch. **5.** 1120 sq. ft. **6.** 24 sq. ft.
7. 4·3602 sq. ch. **8.** 9·888 sq. ch. **9.** $42\frac{1}{2}$ yds.
10. 25·1 ch. **11.** Rs.232 14 annas. **12.** Rs.748 2 annas.

13. Rs.1150 5 annas 10·656 pies. **14.** Rs.398 15 annas 0·48 pies.
15. Rs.25. **16.** 80 yds. **17.** 19 sq. yds. 99 sq. in.
18. 17 ft. ; 9 ft. **19.** 2886 sq. lks.

Examples—VII. B.

20. 324 bighas. **21.** 12 bighas 8 biswas 4 biswansi. **22.** 1080 bighas.
23. 18 bighas 1 biswa 12 biswansi. **24.** $2\frac{1}{3}$ rasi. **25.** $2\frac{12}{17}$ rasi.

Examination Questions—VII.

1. 1764 sq. ft. **2.** 7 ch. 70 lks. **4.** 12,054 sq. ft. **5.** 114 sq. ft.
6. 78 sq. ft. **7.** 7500 sq. ft. **8.** 204 sq. yds. **9.** 125,000 sq. ft.
10. 77 yds. **11.** 2 ac. 2 ro. 8 po. nearly. **12.** 210 sq. yds.
13. 31·12125 ac. **14.** 17·632 . . . ac. **15.** 48·989 nearly ; 844·94 nearly.
16. 10,833 sq. yds. nearly. **17.** 23 ft. ; 27 ft. **18.** 216. **19.** $128\frac{1}{4}$ sq. yds.
20. $3\frac{3}{7}$ ft., $4\frac{4}{7}$ ft. ; or $\frac{4}{7}$ ft., $\frac{16}{21}$ ft. **21.** 1634·99 . . . sq. ft. **22.** 20 ac.
23. 36·9334 ac. **24.** 45·033 . . . sq. ft. **25.** 262·6 sq. ft.
26. 560 sq. ft. **27.** 1229·8 . . . sq. ft. **28.** 52330·3 . . . **29.** 9.
30. $101\frac{23}{27}$ yds. **31.** 4549·92 sq. ft. **32.** 1 ft. ; 2 ft. **33.** 720 sq. yds.
34. $885\frac{1}{3}$ sq. ft. **35.** $13\frac{3}{4}$ ft. **36.** 143 yds.

Examples—VIII. A.

1. 53 sq. ch. 25 sq. lks. **2.** 166·27 . . . sq. in. **3.** 2·7442 . . . ac.
4. 584·56 . . . sq. ft. **5.** 509·22 . . . sq. in. **6.** 618·18 sq. lks.
7. 61·9344 sq. yds. **8.** Rs.274 10 annas 11 pies nearly. **9.** 2519·1 . . . sq. yds.

Examples—VIII. B.

10. 12·9903 . . . biswas. **11.** 19·31 . . . bighas. **12.** 0·62 . . . rasi.

Examination Questions—VIII.

1. 23,636 sq. ft. nearly. **2.** 259·807 sq. ft. **3.** 12 ac. 11 po. nearly.
4. 82·8 . . . sq. ft. **5.** 293·89 sq. in. **6.** $4 : 3\sqrt{3} : 6$.
7. 3·25 yds. nearly. **8.** 36·3 ft. ; 2270·4 sq. ft. **9.** $\sqrt{2}$ ft.
10. 7358·6475 sq. ft. **12.** 0·85 ft. nearly. **13.** $n \cdot \frac{a^2}{4} \cdot \cot\frac{180°}{n}$; 14·5356 sq. ft.
14. $2\sqrt{2}$ sq. ft. **15.** 165·68 . . . **16.** $\frac{3}{16}$ sq. ft.
17. 15·8 . . . yds. ; 9·45 . . . yds. **18.** 649,500 sq. ft. **19.** 2338·26 . . . sq. ft.
20. 1931·3 . . . sq. ft. **21.** 3 sq. ft. **22.** 7·0295 . . . ft. : 119·29 . . . sq. ft.
23. 15·00 . . . ft. **24.** 5625 sq. ft. ; 6495 sq. ft.

Examples—IX. A.

1. 198 ac. **2.** 66 sq. ft. 117 sq. in. **3.** 146·4375 sq. in.
4. 11 sq. yds. 72 sq. in. **5.** 11·904 ac.

Examples—IX. B.

6. 20·16 bighas. **7.** 11 bighas 14·4 biswas.
8. 36·47 bighas.

Examination Questions—IX.

1. 2·1875 ac. ; 500 lks. **2.** 4058·31 sq. yds. **3.** 1602 sq. ft.
4. 192 sq. ft. **5.** 546 sq. ft. ; 13 ft. **6.** 27,813 sq. ft. nearly.
7. 950 sq. yds. **8.** 7230 sq. yds. nearly.

Examination Questions—X.

1. 6·666 sq. ch. **2.** 0 ac. 1 ro. 18·16 po. **3.** 2 ac. 1 ro. 28 po.
4. 13,403 sq. yds. **5.** 1·7967 ac. **6.** 2·05 ac.
7. 0·978 ac. **8.** 0·231 ac. **9.** 132,102 sq. lks.
10. 0·662 ac. **11.** 1·45 ac. **12.** 3·35075 ac.

Examples—XI. A.

1. $10\frac{5}{13}$ in. **2.** 5 ft. **3.** 2 ft. $11\frac{29}{35}$ in. **4.** $11\frac{13}{37}$ ft.
5. $9\frac{17}{37}$ ft. **6.** $733\frac{1}{3}$ yds. **7.** 266·9 mi. **8.** 3·245 in. nearly.
9. $3\frac{1}{5}$ mi. **10.** 180 ft. **11.** $6\frac{3}{7}$ lks. **12.** 230 ft.

Examples—XI. B.

13. 21·315 lathas nearly. **14.** 10·59 rasi nearly.
15. 36·16 lathas nearly. **16.** 7 rasi 3·75 lathas.

Examination Questions—XI.

2. 9 ft. **3.** 26·341 ft. **4.** $33\frac{1}{3}$ ft. **5.** 9 in.
6. $\frac{ah}{a+h}$ **7.** 74 ft. 8 in. **8.** $5\frac{1}{3}$ ft. **10.** 25 ft.
11. 10 ft.; 12 ft. **12.** 8, 6. **13.** $10\sqrt{10}$ ft.; $10\sqrt{15}$ ft.; $250\sqrt{6}$ sq. ft.
14. $\frac{a^2b^2}{(a+b)^2}$. **15.** $352\frac{11}{12}$ sq. ft.

Examples—XII.

Note.—These results, when they depend upon the value of π, are only approximate.

1. $5\frac{1}{2}$ ft. **2.** $58\frac{2}{3}$ yds. **3.** $56\frac{2}{9}$ yds. **4.** 73 ch. 92 lks.
5. 28 in. **6.** $4\frac{2}{3}$ yds. **7.** $21\frac{7}{9}$ yds. **8.** 147 ch. 77 lks.
9. 5500 yds. **10.** Rs.88. **11.** 33 in. **12.** 9 miles an hour.
13. $\frac{7}{44}$ in. **14.** $298,571,428\frac{4}{7}$ mi. **15.** 28 in.; 30 in. **16.** $1272\frac{6}{7}$ in.
17. $27,494\frac{1}{2}$ ft. **18.** 616 sq. yds. **19.** $346\frac{1}{2}$ sq. in. **20.** 4 sq. ft. 40 sq. in.
21. 273 sq. yds. 7 sq. ft. **22.** 20 sq. yds. 106 sq. in. **23.** 55 sq. ch. 4400 sq. lks.
24. 23 sq. po. $11\frac{11}{28}$ sq. yds. **25.** 29 sq. ch. $6211\frac{1}{7}$ sq. lks. **26.** 7 yds.
27. 12·12 in. **28.** 22·13 in. **29.** 0·2213 ch. **30.** 19·7988 ch.
31. 10·6 pies. **32.** Rs.157 3 annas 4 pies. **33.** Rs.$1718\frac{2}{11}$.
34. 10·392 ft. **35.** $50\frac{10}{11}$ ac. **36.** Rs.24 2 annas nearly. **37.** 154 sq. ft.
38. 20·802 in. **39.** 246·66 yds. **40.** 51·84 ft. **41.** 12·409 ft.
42. 378 sq. in. **43.** 26·45 yds. **44.** $\pi : 3\sqrt{3}$. **45.** $\sqrt{\pi} : 2$.

Examination Questions—XII.

Note.—These results, when they depend upon the value of π, are only approximate.

1. 132 ft. **2.** 16·568 . . . ft. **3.** 176 sq. ft. **4.** 39·242 yds.
5. £833 17*s.* 3*d.* **6.** 76·21 yds.; 38·105 yds.; 30·40 yds. **7.** 4840 yds.
8. 10 ft. **9.** 79·5772 yds. **10.** $275,700\frac{6}{7}$ sq. ft. **11.** 3·988 yds.
12. 12,727·27 sq. yds. **13.** 22·50 ft. **14.** 75·42 ft. **15.** 105 yds.
16. $13\frac{4}{7}$ sq. ft. **17.** $3\sqrt{3} : 2\pi$. **18.** 14 ft.
19. $a = b(2 \pm \sqrt{2})$. **20.** 22,008·33 sq. po. **21.** £19 0*s.* $3\frac{3}{4}$*d.*
22. 7·028 yds. nearly. **23.** 5082 in. **24.** 201 sq. ft. nearly.
25. £75 19*s.* 5·544*d.* **26.** 4·9463 in. **28.** $3\frac{6}{11}$ ft.; 39 ft.
29. 281·57 ft. **30.** 5·023 mi. **31.** Rs. 6600.
32. 4·59 in. **33.** $5854\frac{6}{11}$ sq. ft. **34.** $4,320,662\frac{1}{2}$ sq. ft.; 7370 ft.

35. 236 sq. yds. 82$\frac{2}{7}$ sq. in. **36.** 71·06 ft. **37.** 21·46 sq. in.
38. 407·0117 sq. ft. **39.** $\triangle$ = 36 sq. ft.; $\odot$ = 103$\frac{1}{11}$ sq. ft.
40. 27·74 yds. **41.** 15·093 in. **42.** 314·159 sq. ft. **43.** 0·0098 mi.
44. 75·17 ft. **45.** 7 yds. **46.** 11,764$\frac{2}{7}$ sq. ft. **47.** 1·37 sq. ft. nearly.
48. 1·818 ft. **49.** 93·98 sq. in. **50.** 1·3 ft.; 8·168 ft.

Examples—XIII.

1. 32 ft. **2.** 2 ft. 10 in. **3.** 11 ch. 25 lks., or 1 ch. 25 lks.
4. 3 ft. 6 in. **5.** 22 ch. 68 lks. **6.** 7 ft. 1 in. **7.** 15·4 ch.
8. 21 lks. **10.** 9·375 in. **11.** 3 ft. 10·5 in. **12.** 8$\frac{8}{21}$ in.
13. 5 ft. 11$\frac{1}{2}$ in. **14.** 4·4 in. **15.** 105°. **16.** 42°.
17. 26° 15′. **18.** 1960 lks. **19.** 21 yds. **20.** 1 ft. 11 in.
21. 9 ch. 44 lks. **22.** 55·55 in. nearly. **23.** 3 ft. 6$\frac{2}{3}$ in.

Examination Questions—XIII.

1. 13 ft. **2.** 12·6061 in., or 71·3938 in. **3.** 33·8 ft.
4. 10·16 ft.; 7·8102 ft. **5.** 100 ft.; 13·397 ft.; 51·76 ft. **6.** 50·7 ft.
7. 8·74 ft. nearly. **8.** 13·55 ft. nearly. **9.** 4223$\frac{1055}{1056}$ mi.
10. 11·09 in. **11.** 26·321 . . . **12.** 29 ft.; 22·06 ft. nearly.
13. 10 ft.; 50 ft. **14.** 4 ft. **15.** 5·744 . . . ft.
16. 15 ft. nearly. **17.** 125·62 . . . ft. **18.** 14·685 . . . in.
19. 15 ft. 7 in. **20.** 10 in. **21.** 26$\frac{2}{3}$ ft.
22. 6$\sqrt{2}$ ft. **23.** 1·5. **24.** 102 ft. **25.** 314·159 sq. ft.

Examples—XIV.

1. 134$\frac{2}{21}$ sq. in. **2.** 4$\frac{31}{33}$ sq. ft. **3.** 11 sq. yds. 3 sq. ft. 96 sq. in.
4. 20·8828 sq. ch. **5.** 20$\frac{4}{11}$°. **6.** 35° 47$\frac{8}{11}$′. **7.** 4·5126 ft.
8. 7·6752 in. **9.** 210 sq. in. **10.** 71$\frac{1}{2}$ sq. ft. **11.** 5 ft.
12. 6 in. **13.** 6 in. **14.** 1 ft. **15.** 216°.
16. 16·08 . . . sq. in. **17.** 27·46 ft. nearly. **18.** 1008$\frac{1}{3}$ sq. yds.
19. 28·57142 sq. in. **20.** 3·8412 sq. ft. **21.** 2·74 . . . sq. ft.
22. 10·08 sq. ch. **23.** 2042·92 sq. lks. **24.** 19·4917 sq. in.
25. 143·522 . . . sq. ft. **26.** 11·1872 . . . sq. yds. **27.** 1779 sq. ft. nearly.
28. 5·24 sq. ch. nearly.

Examination Questions—XIV.

1. 0·1612 . . . sq. ft. **2.** 12π sq. in. **3.** 1612·5 sq. ft.
4. 160$\frac{5}{12}$ sq. ft. **5.** 908 nearly; 30,520 nearly. **6.** 48·062 . . . yds.
7. 70 sq. ft. **8.** 400 sq. ft. **9.** 8·3 sq. ft. nearly.
10. 14·2698 . . . sq. ft. **11.** 0·3151a^2 nearly (a = side of square).
12. 8326·1. **13.** 2141·2 sq. in. nearly. **14.** 1 in.
15. 22,271 sq. ft. nearly ($\pi = 3·14159$). **16.** 28·4 nearly.
17. 4·77 in. **18.** 1116·114 . . .; 140·521 . . .; 173·637 . . .
19. 44·3 nearly. **20.** 569 nearly. **21.** 218·8 sq. in. nearly.
22. 14·56 nearly. **23.** 55 sq. ft. nearly. **25.** $\frac{1}{2}r^2(\pi - \sqrt{3})$.
26. 27·5 sq. in. nearly. **27.** 56·09 yds. nearly. **28.** 167·279 sq. ft. nearly.
29. 5$\frac{1}{7}$ sq. in. **30.** 1·75 ft.; 0·278 sq. ft. nearly.
31. 208·6 nearly; 100·6 nearly. **32.** 0·06 sq. in. nearly.
33. 1850 sq. ft. nearly. **34.** 28 sq. ft. nearly. **35.** 33·5 nearly.
36. 119·318 sq. ft. **37.** 15·3057 . . . in.; 7·6528 . . . sq. in.
38. 36·32001 sq. ft. **39.** 117$\frac{6}{7}$ sq. ft.

Examples—XV.

1. 6 ft.; 15 ft. **2.** 8 in.; $18\frac{1}{8}$ in. **3.** 58·9285 sq. in.; 235·7142 sq. in.
4. 5·292 yds.; 10·584 yds.

Examination Questions—XV.

1. 519·61 sq. ft. **2.** 125·05. **3.** 26 ft. **4.** 9 in.; $22\frac{1}{2}$ in.
5. $200\frac{31}{36}$ ft. **6** $166\frac{2}{3}$ yds. **7.** 9·01 ft. nearly. **8.** 1·732 ft.
9. 293 ft. **10.** $R\sqrt{3}$. **11.** 6·928 in. **12.** 20 in.
13. 17,084·8 sq. ft. nearly. **14.** 63·8292 . . . sq. ch. **15.** $32\frac{1}{2}$ in.
16. 17·3205 . . . in.

Examples—XVI.

1. 43 sq. ft. **2.** $43\frac{3}{4}$ sq. ft. **3.** $188\frac{2}{3}$ sq. ft. **4.** 11 sq. ft.
5. 182 sq. ft. **6.** 7·605 sq. ft. **7.** 20·2 sq. ft. **8.** 39·3753 sq. ft.
9. 12·44746 sq. ft. **10.** 0·13584 sq. ft. **11.** 6·238 sq. ft. **12.** 360 sq. ft.

Examination Questions—XVI.

1. 2880 sq. ft. **2.** 3168 sq. ft. **3.** 3102 sq. ft.
4. $0·7817 \times r^2$. If $r = 10$, error $= 0·37$. **5.** 2580 sq. ft.
6. 94,000 sq. ft. **7.** $33,093\frac{1}{3}$ sq. ft. **8.** 3900 sq. ft.
10. 924·6 sq. ft. **11.** 392 sq. ft.

Examples—XVII.

1. 242 sq. yds. **2.** 3 in. **3.** 90 sq. in. **4.** 72 sq. in.
5. 1 in. = 264 in. **6.** 1 in. = $\frac{3}{8}$ mi. **7.** 26 ft.; 28 ft.; 30 ft.
8. 12 ft.; 16 ft.; 20 ft. **9.** 6 ft.; 5 ft. 4 in. **10.** 2 ft.; 2 ft. 2 in.
11. 12·726 ft. **12.** 46·18 . . . in.; 65·31 . . . in. **13.** 70 7105 in.
14. 34·641 in.; 48·988 in.

Examination Questions—XVII.

1. 69·28 ft.; 97·97 . . . ft. **2.** 169 : 289 **3.** 42·25 sq. in.
4. $31\frac{1}{2}$ sq. ft.; $94\frac{1}{2}$ sq. ft. **5.** $4\sqrt{2}$ ft. **6.** 1 in. = 1 fur.
7. 1 in. = 69·5701 yds. **8.** 117 ft.; 156 ft.; 195 ft. **9.** 70 ac. 85·6 cents.
10. 6 ft.; 8 ft.; 10 ft. **11.** $10\sqrt{3}$ in.; $10\sqrt{2}$ in.; 10 in. **12.** 28 : 3.
13. 8·049 . . . in. **14.** 1116·9 sq. ft. nearly. **15.** 111·8 ft.
16. 40 ft.; 50 ft.; 20 ft. **17.** 7·46 . . . ac.; 269·4 yds.
18. $4\sqrt{5}$ ft.; $4\sqrt{10}$ ft.; $4\sqrt{15}$ ft.; $4\sqrt{20}$ ft.
19. 10 in.; $10\sqrt{2}$ in.; $10\sqrt{3}$ in.; 20 in. **20.** 376 ft. 2·1 . . . in.
21. 4; 9. **22.** 2549·1 . . . ft. **23.** $\frac{\text{diameter}}{\sqrt{3}}$.
24. 44·58876 in. **25.** $50\sqrt{3}$ ft.; $60\sqrt{3}$ ft.; $80\sqrt{3}$ ft.

Additional Examination Questions.

II.

42. $6\frac{28}{33}$ ac. **43.** 120 ft.; 80 ft. **44.** 50 mins. **45.** Rs.35 7 as.
46. Rs.2869 as. 5 p. 4.

V.

65. 99; 15. **66.** $8\frac{1}{2}$ in. **67.** 1 ac. **68.** 1120 sq. yds.

VI.

15. 2 ft. 1 in.; 2 sq. ft. 48 sq. in. **16.** £120 2*s.* 7*d.*
17. 2400 sq. ft.; 50 ft.; 48 ft.

VII.

37. 3120 sq. ch.
38. $\frac{a+b}{4(a-b)}\sqrt{(a-b+c+d)(c+d-a+b)(a-b-c+d)(a-b+c-d)}$
39. 540 sq. in. **40.** 600 sq. yds. **41.** 49,470 sq. yds.

VIII.

25. 311·25 sq. ft. **26.** 4·682 ft.

IX.

9. 162,463·9375.

XI.

16. 87½ cubits. **17.** 20 in.

XII.

51. 3 in.; 9π sq. in. **52.** 660 ft. **53.** 0·7853. **54.** 1 : 1·0500.

XIII.

26. 19·07 mi. nearly. **27.** $\frac{\sqrt{3}}{2}$ sq. in. **28.** 20 in.; 15 in.
29. 1003 ft. nearly.

THE END

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